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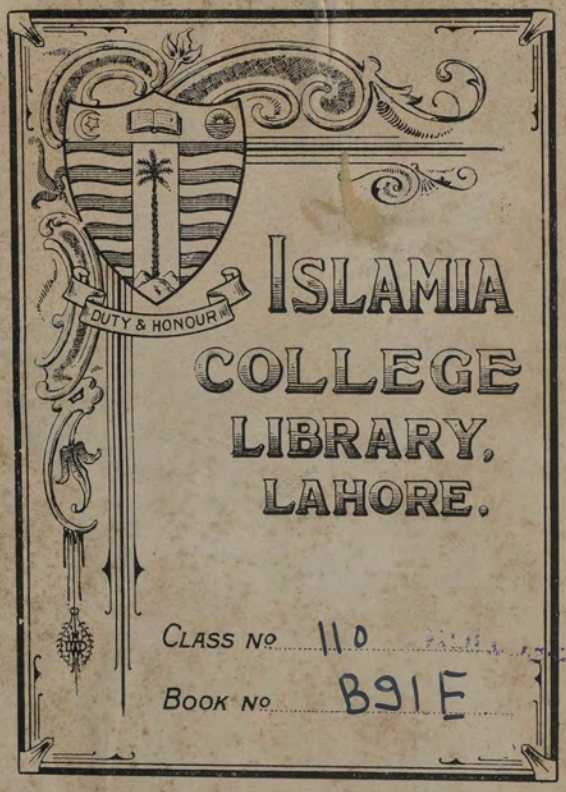

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THE EMERGENCE OF LIFE

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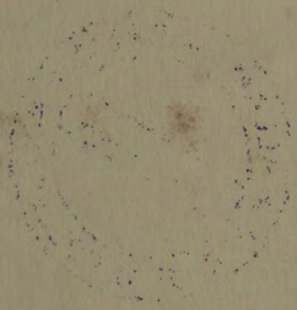

THE EMERGENCE OF LIFE

BEING A TREATISE ON MATHEMATICAL PHILOSOPHY AND SYMBOLIC LOGIC BY WHICH A NEW THEORY OF SPACE AND TIME IS EVOLVED

By JOHN BUTLER BURKE

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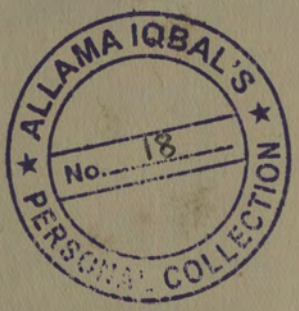

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. . . nature is made better by no mean,
But nature makes that mean; so, over that art,
Which, you say, adds to nature, is an art,
That nature makes. . . . This is an art
Which does mend nature—change it rather;
but
The art itself is nature.

THE WINTER'S TALE.



(Research)

profane in dealing with some of the great and solemn problems under discussion; the language of mathematics being I conceive no less reverent than German, Irish, or Greek; whilst it has the benefit of being universal, pithy, and precise. And though unfamiliar perhaps to many linguists, it is the recognized medium of scientific thought at the present day, as it ought to be the clearest mode of expression of the sublime reason or pure thought in philosophy. I trust that this symbolism will in time supplant all others in the discussion of such problems.

Although the language is essentially that of analysis, investigation, and discovery, the results obtained by this powerful instrument of thought should later be translated into familiar terms when once arrived at. I have endeavoured to do both as far as possible. But I am only too conscious of the limitations of my efforts in both respects. To grapple with such questions may after all be but a vain endeavour; yet the method at least appears to be the right and proper one. My little volume *The Mystery of Life*, a supplement to this, if read conjointly with it in exemplification of the theory here proposed, may serve to elucidate in popular terms what this volume gives in greater detail, as the groundwork of the subject.

JOHN BUTLER BURKE

Villa Thornton,
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Italy.

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be, and as a matter of fact often is, rejected by the referees for the *Philosophical Transactions* of the Royal Society. The latter might in truth be regarded as the proper standard of scientific research and level of scientific knowledge. At present there is no such standard recognized even by the English or Foreign universities among themselves, or in relation to the recognized learned societies in the scientific world. It is evident that Vatican Decrees would hardly be acceptable unless given on scientific grounds.

To be regarded as scientific the view should be accepted by some authorities, recognized as competent to judge, even if it be controversial and not universally acquiesced in. When supposed to be confined to exact knowledge, if it be a matter of controversy, its exactness must obviously to some extent be open to question. This is true of Einstein's theory of relativity, which has become the subject of a heated controversy, particularly in Germany by Lenard and his followers. But no one will deny that it is thoroughly scientific. There is a groundwork of science which is accepted universally; such as the approximate rotundity of the Earth and its rotation round its axis, the Copernican theory of the Universe, the great antiquity of the Earth, the descent of Man from lower forms of animal life, and so forth.

But many facts and theories, on the other hand, though regarded as scientific, are far from being accepted for established truths by universal consent. This does not prevent them from being scientific knowledge.

Furthermore, the universal assent of the scientific world at the present day constitutes no guarantee of finality; and it is not more conclusive than its dissent in the past from much that was once rejected and is now accepted, as indeed all history proves by never-ceasing repetitions.

Euclidean geometry is a case in point. So also is the classical mechanics. And all universally accepted facts and laws of Nature may hereafter admit of modification in the light of more enlightened ideas.

No scientific doctrine has ever been accepted without controversy, and often controversy of a bitter and relentless sort; such indeed as many theologians and ministers of religion, interpreters, and often misinterpreters, of the Gospels have so long adopted as their method.

In scientific matters, however, men of science on the whole now endeavour to show more caution and charity, not to say urbanity, in their criticisms, as they are none the less exacting in their acceptance of facts and principles as established foundation-stones

upon which to build the great superstructure of scientific knowledge. The critics know that they themselves may be brought to book, and their criticisms disproved or discarded as controversial by counter criticisms and rejoinders.

Yet, as we say, the methods employed by such investigators, and particularly by Einstein at the present day, though still debated with intense heat by some authorities, must be regarded as thoroughly scientific. The results obtained, though they might never be established, yet contribute something positive to knowledge by finally presenting a particular view of the subject, as experience gained. And even if these should hereafter be disproved they would but constitute a landmark in the progress of knowledge by the very errors they served to disclose.

What is essential therefore is that the knowledge under consideration should be accepted by some competent and independent authorities of recognized standing and weight of learning and judgement. This, however, does not imply an unchanging and inviolable acceptance. It may be urged that such a view is akin to casuistic Ethics according to which all actions are justifiable so long as a doctor of the Church can be found to defend them. But if it be denied, much that is now regarded as science, particularly in the light of recent developments, not only in psychology, medicine, and biology, but even in the more exact sciences of chemistry, physics, and mathematics, would cease to be reckoned as science in the strict sense of the term. Not only Euclid and Newton, but Einstein would have to go. Newton's work is and always will be science, even when modified, or disproved in the light of more exact knowledge. So is Euclid's, not only historically but actually so. The axioms and postulates of Euclidean geometry may be questioned, but the accuracy of the inferences drawn from those premisses may or may not be questioned, and as such Euclid's contributions to knowledge must be regarded as positive and precise for their day. These may be the fashions of the old clothes of science, the duds which find their way to, and place in, the museums with utility enough to show the development of thought.

More aptly perhaps might these be compared to something like the bone-shaker and the 'ordinary' bicycle from which the 'safety' was evolved; since they illustrate not only the development of an idea, but also the manner in which the various disadvantages and obstacles have been overcome. It is likewise certain that the stage already reached will in turn become obsolete as further improvements come to light. The motor-cycle and the motor-car, as we have them to-day, are not the final works of the creative engineering

mind. There are many types of machines, just as there are many theories, now accepted as sufficiently efficient, that will not so be accepted hereafter as such. And others yet undreamt of will no doubt in time be put forward as more and more in accordance with facts hitherto undiscovered.

Science then consists of that mass of human knowledge which is accepted by some competent judges, past or present, who are universally recognized as competent to judge. Their views may or may not be accepted universally, and indeed may not even be acceptable to the majority at any particular time or place; for their conclusions are not a matter of votes, though the strength of the conviction grows with the number of competent judges who lend it their support. Nevertheless Science necessarily embodies within its sphere facts and theories of a still somewhat debatable nature, sometimes held only by a few; a stage through which most theories have to pass. If for instance we consider the doctrine of spontaneous generation, it will be remembered that Pasteur, Huxley, and Tyndall vacillated from one side to the other of this controversy; and that Huxley in after years was sensitive about it, owing to the patronage he had bestowed upon the creature called Bathybius. But both Huxley and Tyndall always adhered to the belief or 'expectation' that spontaneous generation did take place sometime or another in the remote past.

As we shall see, this view may have been tenable in the seventies of the last century. But nowadays it must be considered crude. The 'expectation' of Huxley's still no doubt holds, but not as regards the spontaneous development of living from absolutely non-living matter. The problem is more subtle than it was at that time supposed to be.

In my book *The Origin of Life* (1906), and in various papers and articles published before and since, amongst them my articles on 'The Radioactivity of Matter', *The Monthly Review*, Nov. 1903, *Fortnightly*, Sept. 1905, and *Knowledge*, Mar. and Apr. 1907, I pleaded for the continuity of vital processes in the gradual scale, from the more complex to the simpler forms of molecular aggregation; adducing reasons for extending the meaning of the word life, so as to include a wider class of phenomena: since the sciences are not divided into watertight compartments, and their divisions are largely arbitrary. There is a continuity between organic and inorganic matter, and there is a discontinuity, like that between the one and the many throughout Nature. Atoms are in the wider sense living cells. Evolution has been continuous, and life in its primitive states must have been different from anything now

observed in Nature. The products of spontaneous generation, if such were possible to-day, would be quite different from anything in the evolutionary series. I have emphasized this again and again; and the ever-recurring criticism, with almost obstinate persistency, has been put forward that the bodies I had obtained were not, and on my own admission could not have been, bacteria at all! That was just my point. I drew a distinction between natural and artificial life.* The artificial cells produced by radium and other substances in the manner described are merely models of the living cell. Living matter itself requires the presence of an unknown substance like radium and yet not radium, nor uranium, nor any radioactive substance known at the time. (See p. 266.) This substance had some of the characteristics of cyanogen (CN) and I designated it at the time as 'biogen'. It was supposed to be radioactive, and most probably a hitherto undiscovered element. It was supposed to be associated with the germplasm or hereditary substance. This 'biogen' was matter in the process of becoming, by condensation of ether-stuff into matter: the chemical elements being the survival and the cast-off products of this evolutionary process; like the planets and solar systems from nebular matter in the now somewhat discredited nebular theory. Thus living and non-living matter, like the organic and the inorganic, are derived

* It has been said with an air of smug self-complacency that my experiments on radium have never been confirmed; and that the bodies I had called 'Radiobes' were doubtless contaminations, due to carelessness, lack of skill, or some trickery on the part of students or assistants or other persons with a perverted sense of humour, who might have had access to my apparatus. In other words that I might have been the victim of a malicious joke, that would no doubt have entitled me to have had the perpetrators thereof prosecuted, and severely punished for their wicked pranks.

I submit that these objections are met by the facts that my results have been not only confirmed by independent observers on the Continent, but that some of these, as almost invariably happens, have actually claimed priority! They were clearly *simulacra* or artefacts resembling Stephan Leduc's structures in his *Mechanism of Life*.

I have referred to these claims in my book *The Origin of Life* and dealt with them accordingly. Further, the fact that the bodies I have described are clearly not bacteria, and could not possibly have been such, renders the suggestion as to contaminations a palpable absurdity! Let nobody therefore say, without betraying a singular incoherency of thought and taint of malice, that no one has confirmed my results in general, that they were due merely to impurities or imperfect sterilization of the tubes used, or contaminations due to unskilful manipulation, or foul play. I repeat that this mode of criticism is neither scientific, philosophic, nor unprejudiced; nay, rather that it reveals a certain perverted disposition and frame of mind, saturated as I might say in the acidity of a cynical asininity, that by its very narrowness and bias must be regarded as incurable as that invincible one-sidedness which it portrays. A review of the situation was admirably given by Dr. Martin Kuckuck of Leningrad in his *Die Lösung des Problems der Urzeugung* (Barth, Leipzig) and also by Horacio Damianovich in *La Doctrina de la Generacion Espontanea* (Buenos Aires).

from the same substance 'biogen' or ether-stuff as two different branches of the tree of reality.

The philosophic aspect of this fascinating conception has carried me away into many regions of thought far removed from physics and biology. I have indeed chosen to hold my peace in fairly silent study and contemplation for I might say close upon twenty years, rather than inflict an over-specialized one-sided view of the subject upon my readers. If there is one thing I have ever endeavoured to avoid it is the doctrine of the universal application of 'leather' to the phenomena of Nature, and to the atom in particular.

We are thus led to consider many of the controversial questions of our time as coming within the sphere or province of Science, though still in the disputed territory of the unconquered or controversial.

The living organism of scientific knowledge is in a state of continual growth and perpetual change. The truth so far attained cannot be regarded as immovably rigid. The functions performed in the process of its growth are as it were more or less constant factors in the course of its development. These also are to some extent subject to change, as capable of improvement. The methods of science are becoming more and more refined, like the instruments used in its operations. And the polished product *emerges* from the sifting and selective process only after the severest treatment which has proved insufficient to annihilate it.

§ 2

What then we are once more compelled to ask is Science? And what is scientific method?

Science has been defined as 'knowledge gained and verified by exact observation and correct thinking'. This does not appear to include scientific theories and hypotheses not yet fully established.

It has also been defined as 'trained and organized common sense'. This applies more to scientific method than to Science itself, which is 'knowledge' of some sort rather than common or uncommon sense. Moreover Science, if common sense excludes all intuitive knowledge and inferences of the 'illative sense', would exclude anything of the nature of a higher sense such as is employed in philosophic speculation. And likewise the mysterious imaginary quantities such as $\sqrt{-1}$, imaginary time, and the abstruse calculations relating to a 'space-time continuum' and a fourth dimension which must surely, though scientific, be regarded as beyond the limits of common sense whether trained or untrained, organized or unorganized. It is, however, largely a

matter of definition, for the latter is perfectly intelligible to a normal mind of a certain level of ability, when sufficiently developed. It cannot be described as 'trained common sense', for there are persons of a quite normal type who though intelligent about ordinary matters, and undoubtedly possessing trained common sense in a very marked degree, could not only never hope to understand these subtleties, but they could not be made to understand what is meant by the equation of a curve.

Thus training, organization, and common sense even when combined do not serve to make up science and scientific method. There must be something more; a power of comprehension and ratiocination with which, alas! common sense cannot always be credited. But in so far as Science appeals to all human beings of a certain level of intelligence, with a certain measure of ability for abstract thought, there is nothing occult in its performances. Common sense must not be made the measure and criterion of sound thinking, but rather should that 'higher intellectual sense' which is based upon reason and knowledge though it be guided by the intuitive power of the understanding through whose concepts or categories it is rendered acceptable. Further, there is the Pure Reason of Transcendental Ideas, to ignore which is to fight shy of these questions.

That the 'sense' must be of a higher order, and at the same time trained, and in some branches of science highly trained, is thus apparent without further emphasis. But as regards the organization, it is in truth here that science, strictly speaking, may be said to enter its own particular province. The organization refers to facts, laws, and theories, with a number of hypotheses not sufficiently confirmed to be regarded as theories. (*a*) The facts must be based upon accurate observation and experiment. They relate either to the facts of the external world as in physical science, or to those of the inner consciousness, as in psychology. (*b*) The laws connecting certain of these facts with other such facts, by the invariable and unconditional relationship of antecedent to consequent are described as cause and effect. (*c*) The methods by which these are established may be those of Deductive or Inductive reasoning, or a combination of both. First by observation and experiment; then induction and generalization or hypotheses; then deduction from these hypotheses or generalizations; and finally observations and experiments on the facts predicted by the deductive reasoning. Every confirmation tends to strengthen the hypotheses arrived at by induction. But a discordant fact, if it does not actually, as a matter of necessity, compel an abandonment

of the hypothesis, would require its modification, by a process of higher generalization and induction, whereby a still more general law may be attained, from which deductions which are again tested by observation and experiment may be made; so that the process is repeated. Higher and higher principles are arrived at. Simpler and simpler laws of wider and wider application become enunciated.

In every case the results must admit of *verification* by trained and competent observers or judges possessing the necessary qualifications as to technical and specialized knowledge. Honesty and, what is rarest of all, *Impartiality* are essential features of the man of science. The latter quality is really extremely difficult to find even amongst the greatest men of science of the present no less than of the past. Honesty of purpose, sincerity of motive, and a scrupulous veracity, exactness, and clearness of thought and expression of a strictly *impersonal* kind are in every case the essential accompaniments of scientific method, as distinct from intuitive, speculative, or transcendental forms of thought. The very plausible results of psychical research which have met with the acceptance of some of the most eminent scientists of our time, in so far as they do not always admit of ready verification, but are usually produced under conditions far from satisfactory, compared with those at the disposal of trained workers in the laboratory, are necessarily precluded from inclusion in the province of exact science, at any rate as they stand at present, except as hypothetical, in which sense they are scientific. The adoption of more rigorous methods, or the discovery of simpler conditions and readier means of verification, with the greater and greater elimination of the personal element, which unfortunately still enters into such investigations, will no doubt in time enable these inquiries to be regarded as more approximately scientific in the exact sense.

The difference between non-scientific (in so far as it is *not immediately verifiable*) investigations, such as those of psychical research, and scientific matter, though it be far from being universally accepted by the scientific world, consists then in the extent and manner in which the results may be tested and verified without bias. [Fraud, conscious or unconscious, and magic such as an accomplished conjurer might perform, have to be eliminated. Unconscious fraud is the most difficult to disprove owing to the recognized and palpable honesty of the observers.]

Most scientists as ordinary mortals have 'an axe to grind'. They may even form themselves into groups, or 'schools of thought'; earnest but prejudiced adherents and advocates of certain views, or irreconcilable opponents of others. Their opinions are therefore

tinged with prejudice. Examples of all sorts may be found in the history of science. From the Copernican theory and the Galilean and Newtonian conceptions of force and mass, to the Einstein general theory of relativity, controversy and personal predilections, on one side or the other, are throughout apparent. Weismann's germplasm theory and the non-transmission of acquired characteristics is perhaps the most remarkable case in point. In this the overwhelming weight of expert opinion was for two decades and more uncompromisingly on the side of Weismann. Only gradually is equilibrium being regained, and a higher generalization slowly formed, that transmission does take place but at an extremely slow rate. In all advancement there is at work the Hegelian method of dialectic. From the negation of opposites, a higher synthesis is formed, constituting a greater approximation to the truth.

We shall enter more fully into this question, and its validity, when dealing with the philosophic aspect of the fundamental problems with which we shall have to deal.

It will be admitted, however, that an accurate knowledge of facts based upon observation and experiment, and a well-disciplined reasoning upon such facts must on the one hand constitute an intelligence of more than ordinary common sense; and on the other hand while Science and common sense have one strong conviction or prejudice in common, that is, as to the reality of the external world—a conviction in which they join issue with the idealistic metaphysicians, who consider that they have attained to a still higher synthesis, in denying the existence of a world independent of the self. Science is content to assume, as a matter of induction, that the external world is a permanent reality. This is the fundamental hypothesis in all physical science. It is in fact a mere working hypothesis, by no means essential to the explanations of science itself, as has been recognized by many philosophers. The belief in the external world, though not necessarily true, has formed throughout the basis of all scientific thought and work, as well as of our ordinary everyday life and actions.

The value of this inductive reasoning is often overlooked even by those who have ventured to give laws to philosophy. The metaphysician in his quest for certainty is for ever in difficulties, owing to the fact that although he bases his knowledge on experience, he seldom contents himself with the hypothesis and generalizations of induction; but looks for absolute certainty where none may be attainable beyond that of his own consciousness. As the basis of our reasoning faculties, such certainty as is confined to actual experience, thus limiting our knowledge to the sphere of

our own individual consciousness, gives the limits of personal knowledge. This, however, does not prevent our going beyond actual experience, by a process of induction, the probability of the conclusions of which fit in with the notions of common sense, as simple and practical generalizations. The theory of relativity appears to have removed the notion of force as an ultimate conception in science, as it was regarded in the Newtonian mechanics, and has reduced it to a necessary result of relative motion, by what is known now as the 'principle of equivalence'. From this it is not a very great or difficult jump to some form of idealism. It will be seen that the problems dealt with in this book demand a higher generalization than that which common sense, or even the more than common, and higher sense of the classical science of the nineteenth century required, and that something still higher and more refined is not only needed but attainable, in a manner to be expected and hardly to be denied.

§ 3

Whatever view one may take of the nature of reality, the importance of scientific speculation, and the significant part that hypotheses play in the investigation and discovery of knowledge cannot be questioned. It is usually expected that the results of such investigations should be 'finished products'; final answers to the questions with which they are supposed to deal. But the results of scientific research are never really finished products, never final answers, however much they may approximate to the final ideal and the ultimate conception of truth.

The principle of continuity in nature, coupled with that of causation with which it is intimately associated, and the belief in the external world as an independent reality, are three articles of scientific faith without which, in the opinion of most scientists—though as we have seen not necessarily of all—scientific inquiries could hardly be conducted. These are merely 'expectations' based upon a degree of *moral certainty*, the basis of all inductive reasoning; strengthened and confirmed, day by day as our experience grows, as matters of probability, admitting of modification and improvement in the light of more advanced knowledge. The degree of probability for the individual thinker depends upon his knowledge and experience, his predilections, his temperament, and the general constitution of his mind. Like the man in the street, the man in the laboratory is then for practical purposes content to base his work upon the generalization of such a foundation. For practical purposes indeed induction, and the doctrine of chance, or

probability, form the basis of all sane thought, when coupled with the sense of the limitations, the boundaries of knowledge and experience naturally impose. Thus the practical reason as distinct from the pure reason, as Kant would have put it, the belief in the 'thing-in-itself' may be regarded as the basis of the 'transcendental' philosophy; the 'noumenon' as distinct from the 'phenomenon'. Later systems of thought have endeavoured to dispense with the noumenon as a necessary condition of the phenomenon, but the critical philosophy would not hold good without it.

Kant was not blind to the defects of the idealism of Hume though it led up to his own. We shall see that the modern 'Idealistic-Realism' is but a modification of Kant's doctrine; and that the nature of thought and perception when properly analysed, necessarily implies a basis independent of the self and yet of the nature of an idea. That the world though a reality independent of the individual, nay of all finite individuals, is none the less a world of thought: a world in which the idea whether in thought, in action, or in feeling, that is in cognition, in conation, or in affection, forms an inseparable unity as the very essence and condition of reality.

For purposes of investigation then the methods and axioms of science are of great though not perhaps of paramount importance from the philosophic aspect. The world may have to be turned inside out, from the physical as well as from the political standpoint. But it must be by no Bolshevist revolutions. The groundwork of science can give way only gradually to the reforming transformations. We cannot regard knowledge as kaleidoscopic. Yet the scientific methods of the empirical schools having worked out wonders since the time of Bacon, have, however, left us bereft of any comprehension of the basis of knowledge; and notwithstanding the stupendous efforts of the critical philosophy of Kant, we are still much where we were as regards the problems of life and mind, unless some radical conceptions be accepted. His transcendental philosophy, however, is the key to the solution of the problem.

The investigation of Nature by purely empirical and experimental means has led to results that the older methods had completely failed to arrive at, and were in fact unfitted to pursue. But between this and newer conceptions of the categories, particularly the forms of intuition of Space and Time, there is no necessary contradiction. The methods and results are not only not contradictions nor even contraries, but other alternatives than the most obvious are often possible; and as it appears plausible and useful.

The whole tenor of the new method lies in the exhaustive examination of such apparently exclusive classes of ideas according to the analysis which mathematical reasoning, rather than the purely logical ratiocination, so powerfully and lucidly calls forth.

§ 4

The universal science of mathematics, of which logic nowadays is regarded as a special branch, ought to be considered, and will no doubt in time be regarded as the universal language of clear thought. For as sterility in results in thought as in all else breeds discontent though nothing more as a contradiction, so does fecundity, particularly in ideas, arouse the enthusiasm of every intelligent being. And the methods of analysis here employed, by their fertility, open up new vistas and new paths for the disheartened investigator to pursue.

To Leibnitz I might almost say I owe the inspiration of the conception of this somewhat ambitious project, that is his almost chimerical attempt at a universal logic by which all problems might be exhaustively dealt with. Such a project of so enticing a description had in my early youth a fascination for me and an overwhelming boyish attraction.

My ambition was if possible to evolve some method by which the problems of philosophy, and other apparently inexact branches of knowledge, might be dealt with, with precision by the methods of mathematical analysis and rigorous demonstration. It would seem that the scheme initiated and sketched out by Leibnitz, though never developed by him, except as regards the infinitesimal calculus, is perhaps the germ of subsequent ideas and investigations in this direction. As Sir Joseph Larmor, the FitzGerald of the present time, has well said, 'Leibnitz always got to the basis of everything'.* *La Logique de Leibnitz*, by Louis Coutarat, gives a learned, lucid, and detailed description of Leibnitz's ideas of this universal method and the stage to which they had reached.

Descartes' method as outlined in his *Discours sur la Méthode* and in his *Méditations* was an attempt to deal with these problems in a scientific manner, and by strict ratiocinative or deductive thought. Spinoza proceeds with similar rigour, and perhaps equal ineffectiveness, although the closely reasoned system of thought of the latter is unequalled in the history of philosophy; for the beauty and elegance of the demonstrations, which like Euclid's 'Elements' are not only consistent but true, provided the contradictory be absurd.

* At a meeting of the Mind Association, Aristotelian Society, and British Psychological Society held at Bedford College, 1919.

Descartes' invention of analytical geometry in the application of algebraic equations to the solution of geometrical problems by the aid of the conception of coordinates marks an epoch in the history of science. Leibnitz's method, however, was quite different. He dealt with classes or groups of ideas, which he endeavoured to analyse and arrange according to logical laws of thought in an algebraic fashion. This universal language of analysis he hoped to make as a matter of clearness, precision, and conciseness the language of the learned, which would essentially form a part of the mental equipment of every educated man.

Loose reasoning from empirical foundations, and inaccurate habits of thought, should be regarded as offensive as bad grammar and bad syntax. Until people know exactly what they mean by the terms they employ and the propositions they assert; until they are able to formulate general laws based upon facts of scientific experience, and not mere haphazard guesses supposed to rest upon observation of an equally casual sort; until they learn to draw rigorous conclusions from them; or at least, when such is not possible, the most probable conclusions, with a knowledge of the limits of accuracy and probability, they should not be deemed to have grasped the full import of their assertions. When a statement is soberly and seriously made that a certain thing, idea, or event is dependent upon another thing, idea, or event, one cannot be said to know very much about it, until a 'function' connecting one with the other is ascertained; until a curve can be drawn representing their relationship; in other words a formula be expressed giving the relationship of one to the other. Neither Descartes nor Leibnitz had developed his ideas sufficiently to obtain this combination about things generally as he desired.

Leibnitz complains bitterly of the discouragement he received from his scientific friends, with the exception of a few enlightened persons such as his compatriot Oldenburg, the Secretary of the Royal Society. This discouragement appears to have mitigated the ardour and hindered the success of his pursuit. Even the symbolic treatment of Newton's theory of fluxions met with the cold indifference and chilling scepticism of no less enlightened a critic than Huygens, at whose feet he had sat in Paris as a disciple in his earlier days. It is to Huygens's credit, however, that having done all the injury he could by unsympathetic discouragement of Leibnitz's methods, he subsequently recanted his opposition and became an ardent supporter of his views, nay almost in turn his disciple. This proof of magnanimity added to his reputation as it took away from the shallow prestige of the group of smaller men around him.

Obviously if a piece of reasoning is coherent and exact it ought to admit of being expressed concisely in a mathematical form; by which not only any possible flaw could be exposed, but further developments from its rectification readily deduced. Much of the everlasting quibbling in most subjects, including economics, may be attributed to the unsatisfactory statement of laws and facts, with confused reasoning upon them, in ordinary language which is inadequate to express the full relationships involved. It is easier to lay the finger on a fallacy when it is expressed analytically than when it is involved in a verbiage of high-sounding phrases. On the other hand, it may be said that the statement of an argument in mathematical form often only serves to complicate matters, by obscuring the contents without clarifying the issue. This may appear to be quite true to the uninitiated; but the solution of intricate problems in the hands of experts trained in that language is unquestionably facilitated, in some cases even rendered possible, through the simplification afforded by the use of algebraical notation. For purposes of popular comprehension it must be translated. It is not to be expected that in general conversation, or in popular discourse, anything so pedantic as a rigorous scientific method is to be observed, by persons possessing the sense of humour, or a vestige of common sense, still more that higher sense that implies the sense of proportion, as well as the balance of all the intellectual faculties of mature judgement, perception, and reflection. Opinions and convictions for which no reasons need or could adequately be given, require the workings of the 'illative sense', and will no doubt continue to be held and to be expressed, with all due consideration and regard for those of others. But in serious discussions, the methods of exact science will in time it is hoped become the recognized modes and canons of discourse, not only amongst the learned, but amongst the less highly educated and intelligent in all classes of the community.

Such mathematical analysis should be employed only as an aid to intricate calculations or involved reasoning. The part that instinct and intuition play in the rational operations of thought cannot be overlooked as purely natural processes. Yet all sound argumentation must admit of being analysed when desirable by logical methods, and in extremely complex problems the assistance offered by mathematical treatment may actually be indispensable.

If any doubt exists upon this point, one need only take up the study of the theory of relativity, quantum and wave-mechanics, or the theory of groups, to convince oneself of the truth of the assertion. Such results as have been reached could never have been

obtained by the unaided intelligence, any more than the secrets of minute biological structures could have been detected by the naked eye, without the use of the microscope; or of atomic structures without the use of the Röntgen rays; and on a vaster scale, the structure and immensities of the Universe without the help of the telescope, the spectroscope, and the interferometer.

Mathematical reasoning is not confined to the ideas of *Quantity*; it can also be applied to those of *Quality*, substance, and the other categories; for its operations are in fact on classes and relations, and not necessarily on numbers which form a particular class of classes.

§ 5

Leibnitz's *Universal Characteristic* was considered, by himself at least, to render an exhaustive treatment of such problems possible. He regarded this discovery, prematurely no doubt, as the greatest of his age; although as we have seen it was not so regarded by his contemporaries. He maintained that by the introduction of a universal symbolism of thought with definite laws or rules for the combinations of classes or their elimination, as in mathematics—a method that would be intelligible in all languages—the truth or falsehood of any argument or reasoning process would depend upon the accuracy or error in calculation; leaving no room for mere loosely formed opinions, wild guesses, or immature and undeveloped judgements in serious matters.

In the main, the analysis and reduction of ideas to a few axiomatic and unanalysable concepts, the classification of these, together with the study of their consistent groupings in classes, may be said to constitute the basis of his method.

As W. R. Sorley remarks:

'The logic of Leibnitz is an art of discovery as well as of proof; and as such, applies both to the sphere of reasoning and to that of fact. In the former it has by attention to render explicit what is otherwise only implicit; and by the intellect to introduce order into the *a priori* truths of reason, so that one may follow from another, and they may constitute together a *mode intellectuelle*. To this act of orderly combination, Leibnitz attached the greatest importance, and to it one of his earliest writings was devoted.

Similarly in the sphere of experience, it is the business of the art of discovery to find out and classify the primitive facts or data, referring every other fact to them as its sufficient reason, so that new truths of experience may be brought to light.' (Article on 'Leibnitz', *Encyc. Brit.*, 11th edition.)

Leibnitz's enthusiasm for these ideals and methods, which he styles 'l'alphabet des pensées humaines', was perhaps more intense

than the sense of satisfaction which their realization ever afforded him: since he perceived in time that the knowledge so far attained, together with its classification at the period in which he lived, was indeed, when taken seriously, quite hopeless; certainly for any one person to endeavour to attain in any special subject. He sought, but sought in vain, the co-operation of learned societies such as the Royal Society and the Académie des Sciences; but with bitter and pathetic complaints he tells us of the sarcasm used by his contemporaries, who he sometimes forgot were keen and zealous if not always jealous rivals, with their own axe to grind; not a few of whom would gladly have buried his in everlasting oblivion if they could. In this he betrayed a lack of that sense of humour which is relieved by such admissions as the following. In the *Nouveaux Essais*, IV. vii, § 11, he relates: 'On montra à Casaubon la salle de la Sorbonne, et on lui dit: "Voici un lieu où l'on a disputé durant tant des siècles!" Il répondit: "Qu'y a-t-on conclu?"'

It might indeed be remarked that previous to Galileo or Newton a Casaubon might well have asked at any laboratory, 'Qu'y a-t-on conclu?' The answer would have been different from that which he could have received to-day. In fact at each stage of the world's progress the same question could have been asked with equal effectiveness or ineffectiveness, according to the subject-matter to which the question actually referred. What was unknown has become obvious, and what is still unsuspected will in time become familiar fact. But that so little progress has so far been made in some branches of knowledge, such as philosophy, metaphysics, or even 'political science', does not mean that these subjects are destined ever to remain obscure, undeveloped, and inexact. In every case when little hope of progress is apparent and a cul-de-sac has obviously been reached, new methods and new instruments of thought, a new idea or a fresh perspective of the situation from a loftier standpoint, may afford a glance of hope and a potion for salvation. Ideas of this sort are often atavistic in their origin and unexpected in their appearance; not being subject to the ordinary laws of development of knowledge, in gradual steps, through the slow growth by small increments of continuous variations. The abrupt reversal to some forgotten notion long since abandoned to the dismal limbo of the past, the 'new' idea emerges, and as truth itself which 'in time will out', from the mass of unrealized material along the history of thought, it emerges to perform a radical change. Not, however, in the true sense in which 'emergence' is used later.

Even in economic science, which is already recognized to some

extent as a branch of exact science, discussions will ultimately turn not only upon statistics and their degree of accuracy, but upon the mathematical analysis of the facts by the theories of probability.

Taylor's theorem, Fourier's analysis, and the like will enter into the elucidation of many causes underlying economic events, as they do so already in meteorological phenomena. We know this science once baffled the casual observer by the complexity of its effects though resembling economics in many ways. The two are governed by strict laws. Students of economics like Jevons, Marshall, Edgeworth, Keynes, and others at the present day already adopt, as far as possible, a rigorous scientific method in the discussion of such questions. But, in general, opinions are formed in a most superficial manner, even by those professing expert knowledge. The day, however, is not so very far distant when, as I venture to think, statesmen and even politicians may be forced, from the standard of knowledge, to reduce the solution of a complicated economical or political question, as one would an engineering or other scientific problem, to the solution of a system of equations, and the differentiation or integration of an infinitesimal series as in the differential and integral calculus!

In philosophy such apparently unanswerable questions as Herbert Spencer's infinite series of 'the explanation of an explanation, of an explanation, of an explanation *ad infinitum*' leading as he thought to the *unknowable*, may in reality be found to be something of the nature of a 'converging series', and therefore finite; [somewhat, though not quite like, Achilles' pursuit of the tortoise], the classes or ideas merging into a continuity with a finite conception of unity underlying them, as the continuity between the one and the many; the final or ultimate problem of philosophy. This problem of the one and the many, or that of truth itself and the ultimate basis of knowledge, turns upon the relationship of classes of ideas expressible in a system of equations, and the overlapping of classes.

§ 6

It was not till after the lapse of more than two hundred years that the matter appears to have been once more fruitfully taken up by George Boole, Professor of Mathematics at Queen's College, Cork.*

* *The Mathematical Analysis of Logic.* 'Being an Essay towards a calculus of Deductive reasoning' by George Boole (1847).

An Investigation of 'The Laws of Thought', on which are founded The Mathematical Theories of Logic and Probabilities, by George Boole, LL.D. (1854). New edition by Philip E. B. Jourdain: The Open Court Co., Chicago and London, 1916. Russell says in *Mysticism and Logic*, p. 74, that 'Pure mathematics was

There is little evidence, if any, in Boole's writings of his acquaintance with Leibnitz's logic. His ideas appear to be quite original, having obviously been developed on quite independent lines. Boole's method, though cumbersome at times, is the basis of all subsequent investigations on the subject; and merits on that account the particular attention of all serious students of logic, and especially those of the mathematics of logic.

The system met with severe treatment at the hands of W. Stanley Jevons, who evolved a symbolic logic on the fundamental ideas then initiated. Some important improvements were later suggested by that great scholar the late J. Venn, and by J. N. Keynes; whilst more recently A. N. Whitehead and Lord Russell in England, and G. Peano and C. Burali-Forti in Italy, have developed entirely new conceptions of the mathematics of logic. The compiler and critic of Leibnitz's logic, Monsieur Louis Coutarat, already referred to, has himself made some valuable contributions to the subject, as given in his *Algèbre de Logique*. A bibliography of this particular branch of the subject will be found in his book on Leibnitz [loc. cit.].*

In the system which I have evolved I have endeavoured as far as possible to adhere to Boole's method, being the more directly related to the familiar Algebra than the systems of Coutarat and others of recent date, and more easily manipulated for abstruse inferences.

Boole's method is in principle the basis of all reasoning; in which an exhaustive enumeration of all possibilities should be considered and subsequently some eliminated, when incoherency arises. In ordinary reasoning, as in geometry or chemical analysis, there is classification and selection; proceeding step by step by the elimination of the irrelevant. In algebraic analysis the same applies, but the intelligence and skill lie more in the correct use of the equations, their solution and interpretation; which often require a high degree of ingenuity. It is indeed in many complicated problems vastly more powerful than the ordinary unaided reason, both as regards the method of attack and the results obtained; at times indeed it

discovered by Boole, in a work which he called the *Laws of Thought* (1854). This book abounds in asseverations that it is not mathematical, the fact being that Boole was too modest to suppose his book the first ever written on Mathematics. This may be an exaggeration, but there is much truth in the appreciation, which is both generous and just.

* During the past few years valuable work has been done by A. Fraenkel, *Einleitung in die Mengenlehre* (1928); and by Rudolf Carnap, *Abriss der Logistik* (1929), a good abridgement of *Principia Mathematica* and other recent works upon the subject. W. E. Johnson's *Logic* and Miss L. S. Stebbing's *Introduction to Modern Logic* should also be referred to, and F. P. Ramsey's *Foundation of Mathematics*.

would appear as if almost miraculously so. Usually, as in the system of Jevons and Keynes, all the possible combinations are written down and those inconsistent with the premisses then struck out: a process which can easily be effected when only a few classes are to be dealt with. This in substance was the basis of Boole's system. But the method of elimination and reduction of equations enables one in most cases to draw an inference with regard to any particular class without the labour involved in this full expansion, by solving a system of simple simultaneous equations, usually of a concise type. This he does with much lucidity and skill in the analysis of some of Plato's arguments, and also Clarke and Spinoza's; a point that has been much overlooked by his critics.

It is true of course that many of Boole's calculations are almost hopelessly, and equally unnecessarily, complicated; admitting of simple solutions by quite familiar means. But this would apply to all analysis. Many geometrical problems can be solved much more easily by pure than by analytical methods; but others could never have been tackled except by the aid of some algebraic calculus. Quaternions may be regarded as a case in point. They have been said to be 'more curious than useful'. But vector analysis, the direct outcome of quaternions, and only a simplified form of that calculus, is to-day an indispensable attendant of all higher physical research.

There can be no doubt that some of the fallacies in Clarke and Spinoza's arguments are easily brought out by Boole's analysis. The method in some cases may be extremely laborious and indeed may require great patience and skill. If, however, all really complicated, and in some cases comparatively simple questions; if all philosophic and particularly metaphysical discussions were forced through the mill of this analytical treatment, we should soon have an end of much fallacious philosophy, and be compelled to limit ourselves to the clearest and most exact reasoning, knowing at least at a glance if we differed, where we differed, and why.

§ 7

A great difference exists between this system of logic and the Scholastic or Aristotelian. The latter gives only particular solutions. But the former offers a general solution giving every possible view of the question, sometimes no doubt tediously so to the unexperienced worker in this curious field of research. It is, however, the basis of all exact reasoning. It affords an exhaustive classification of all classes and possibilities of ideas; whilst the

power and elegance of the method lies in its applicability to many problems of probability to which the simplified plans of Jevons and his followers would hardly have extended. This is a point that Jevons entirely overlooks.

The method of analysis can be applied to the dissection of the most complicated argument. Every possible answer is presented to a given set of premisses.

According to the Aristotelian logic it is not possible to draw any conclusion from two negative premisses. For instance, from the two propositions, that (1) No men are infallible, (2) No men are in their bodies immortal; obviously no conclusion follows by the ordinary logic. But by Boole's method the inferences of the following nature do appear, i.e. (1) All men are neither infallible nor bodily immortal. (2) If a being or thing is bodily immortal or infallible it is not a man. (3) Anything that is not a man, for anything we know to the contrary, may or may not be infallible, or bodily immortal; and the latter remark applies equally to beings that may have no bodies at all. The subtlety, wit, and refinement of the Boolean algebraic solution of the question is thus apparent. It never loses sight of the classes with which we may not at the moment be concerned, and might on that account be inclined to overlook. When we are dealing with men we are apt to forget the possibility of angels; and when we think of man as a mere fallible mortal bodily creature of ephemeral worth and transitory conception, we are apt to overlook the possibility of his being what the premisses do not deny, a spiritual being, at once immortal, capable of attaining, at times at least, a clear perception of the truth; and certain that as regards his knowledge of himself as the basis of his certainty he knows not only that he is, but may for ever be.

All this is passed over with indifference by the Aristotelian logic, as something quite irrelevant to the premisses. In a sense it is irrelevant; but only in a sense. As in the Hegelian dialectic the Boolean method of expansion is for ever affirming and denying the most subtle distinctions almost *ad infinitum*. And although it differs from the Hegelian method in starting as Aristotle does with the law of contradiction; yet as we shall see this is by no means essential to its purpose. Two contradictory or contrary statements, neither of which is absolutely true, may not be incompatible. So that by contrary or contradiction we may be able to arrive at a higher synthesis having a closer approximation to the truth. And this again not being absolutely true can by the aid of its contrary or contradictory, as the case may be, lead to a still higher approximation to the truth. And so on until we reach the Absolute. This

is the Hegelian method, upon which Boole himself pours scorn. But only because he did not understand Hegel, who was then still new to him, nor did he perceive with what beauty the method he had initiated leads to an elegant treatment of the fundamental difficulty in the great problem of philosophy, the problem of the One and the Many.* The law of non-contradiction applies only to a particular 'event'; at a particular time and at a particular place. In other words it is 'static' thought. In reality all states of consciousness are 'dynamic' and as such in a perpetual state of change, so that the law of non-contradiction does not in fact apply. Change implies transition, and the period or stage of transition must necessarily be one of contradiction, in which both or neither of two stages may be said to exist, which is an obvious contradiction although both statements are approximately true, and neither is strictly accurate except for the instant. Here the new method comes to our aid. Obviously continuity and the law of non-contradiction are contradictory! Boole works out his system algebraically on the assumption that the law of non-contradiction holds. But as we shall see it is by no means essential that this law should hold; and experience shows that in the world of reality it does not hold except for 'static' events. Hence a mathematical logic can be evolved in which room for continuity may be found. This includes the Hegelian logic and a great deal more.†

I have endeavoured to do this in the following pages, never forgetting that the symbolism represents in its general form that 'higher sense' which *dynamic thought* as distinct from *static thought* implies. And yet to be but just and fair to the latter it acts on the promptings of the former. 'Common sense' must be regarded as higher than static thought; the thought of logicians since the time of Aristotle. The Pure Reason and not the so-called Understanding in Kant's sense is the Higher Sense of Mankind.

§ 8

Having devoted many years to the study of the various methods that have been employed for the analysis of logical processes by algebraic means I believe that I have succeeded not only in simpli-

* Whitehead and Russell's theory of Types, Classes that do not overlap, is considered elsewhere. A class cannot be identified with one or any of its members. Hence the One and the Many are not identical, but different 'types'; from A is B and all B's are C's it does not follow that A is C, if B is used collectively in one and distributively in the other premiss as Peano insists. There are really two Middle terms.

† Whitehead and Russell regard an instant, or event, as points in the time series and that there is a one-one relationship between these and the points in a spatial line. Hence motion becomes possible, and Zeno's paradox is explained.

fying Boole's method, but in evolving an entirely new system of logic, that gives a novel conception of the world and of thought, as partially viewed by Hegel, but much advanced in the light of more recent conceptions, such as the theory of relativity and the theory of groups require.

So far there have been two kinds of relativity, the relativity of motion, space and time, and the relativity of knowledge. Yet there may be absolute motion, absolute space, and absolute time; just as there may be absolute knowledge. But these would belong to static as distinct from dynamic thought. All the antinomies of Kant are due to this difference in the two kinds of logic or modes of human thought.

Unfortunately the mathematical method does not lend itself readily to popular exposition, and its value lies in the aid rendered to the investigator himself, as distinct from his audience. The value of the method is in the analysis; the dissection, and subsequent synthesis of a particular problem as a preliminary to the translation into words of the whole argument for popular consumption. All classes are included within others, *ad infinitum* when the imaginary $\pm\sqrt{-1}$ is introduced. In the two extremes, however, we have on the one hand that the ultimate class includes no other; and on the other hand that the ultimate class can be included by no other. It is evident that if we regard such classes as arranged in a definite order, one within the other; then the ultimate classes may be represented as at $+\infty$, and $-\infty$, respectively or vice versa in the ordinary way with numerical quantities.

When the idea of order in the arrangement of classes is introduced the Gaussian system of coordinates may be employed with advantage in a manner enabling us to represent certain relationships without definite units of measurement in the Euclidean sense. All that is required is the order of succession of such classes, or events. This important extension of the conception of Cartesian coordinates enables mathematical analysis to be employed when *quality* rather than *quantity* is the subject of investigation. The logical connexion between two sets of facts as regards *quality*, *substance*, *action*, *passion*, and the other categories can be represented, as well as those of *quantity* in *space* and *time*, by definite classes in a given order, although the actual units are not defined or definable. All reasoning implies a synthesis in the unity of identity and difference of types in the synthesis of a judgement.

This extension of the idea of coordinates, generalized by Gauss and Riemann, has enabled Einstein to work out his now famous general theory of relativity and gravitation on the general principle

of the relativity of motion by the aid of Minkowski's conception of a space and time continuum in four dimensions.

What is required is the order of succession of the classes of events. It is applicable in general, as we have said to all classes of ideas.

In the Gaussian system of coordinates, the relationships are expressible without a definite unit of measurement in the Euclidean sense, all that is required being the order of succession of classes, whether in space or time or in regard to any other category, along which the order of classification is made.

The method can be applied to all logical propositions in which there is a 'function' connecting two or more variables. It is easy to see that curves may be drawn representing the relationship between the various grades in any two sets or series of classes; without any idea whatever of absolute units of measurements, the order being all that enters into the calculation. Classes and individuals do *algebraically* overlap unless found incompatible.

Again the method may be applied in psychology in correlating different grades of general intelligence represented in an arbitrary scale indicative of certain grades or classes. A similar method might be used to represent the relationship between perception and the intensity of emotion, indicated by certain physiological tests, such as the electrical resistance of the hand as a measure of emotion. In the latter case no doubt one variable, the electrical resistance of the hand, does admit of measurement in terms of definite physical units; but, as we shall see, the modern theory of relativity renders such units in physics ultimately relative. In any case the intensity of the emotions can be estimated only by one being greater than another; and a third being greater than one and less than another. There is thus a continuous scale of gradation established; and as in the theory of relativity we can make what choice of units we please, according to the particular circumstances and conditions of the problem.

§ 9

It is of interest to note that the late Professor F. Y. Edgeworth of Oxford, who was at one time a Classical Scholar of Trinity College, Dublin, before going to Oxford, where he took up the study of mathematics and economics, was amongst the first, if not the first, to show that economic problems lend themselves readily to rigorous mathematical treatment.* The principle of least action

* *Economia Politica* by V. Pareto of Lausanne (1898) is an important contribution to the subject treated mathematically. He bases his work largely on Edgeworth's *Mathematical Psychics*.

leads to the most probable course of events: just as in the theory of relativity.* That the laws of Nature are themselves the most probable manner in which Nature should act, is a necessary consequence of this fundamental physical principle.

On the other hand the theory of numbers, as has been said, is like the Aristotelian logic merely a branch of the larger and wider science of mathematical logic. So that the theory of groups, at any rate in part, is not without its bearing upon the phenomena of Nature. A recurring decimal in the theory of numbers has its counterpart in a more general manner in the logical theory of groups of classes. And a recurring series is conceivable in one as in the other.

The relationship of subject and object constitutes a connexion of an inviolable nature in the manner indicated.

The relationship of subject and object, *S* is *O*, implies that both terms or ideas may be developed in an apparently endless series of relationships. For the object must itself be the subject of some other idea, and so on *ad infinitum*. Likewise with the subject in the first instance. It is the predicate of some other subject, and so on *ad infinitum*. But not really so in every case; for it is conceivable that subject and object might ultimately become one, or return into themselves. This is not recognized in the theory of Types which ignores self-consciousness altogether.

And in the latter case there would be a perpetual repetition of the process, as in the case of the growth and reproduction of a living organism, in which the seed can apparently repeat itself for ever, as in a repeating or recurring decimal. As the apparently endless progression of an idea in space and time involves a 'curvature' and returns into itself; so any series of ideas in endless progression alternately of subject and object will ultimately become one with itself; or else progress onwards with ever-recurring repetition of its signal characteristics. As the seed develops into the plant, its branches, and its leaves and blossom and fruit, into the seed again from which it arose; and the seed once more when resown again develops into another such plant, and another and another and so on; so the recurring decimal, in however complicated a form we may choose to give it, even after many million places of decimals, can as a *group* of figures recur again and again: so reproduction occurs as though the substance of living matter were inexhaustible; and the tree of knowledge or ideas develops into countless forms of apparently inexplicable but in fact intelligible ramifications.

* A. S. Eddington's *Mathematical Theory of Relativity*.

It is thus from the properties of ideas, as in the case of the theory of numbers, that knowledge and nature itself, one class of knowledge, are evolved; by the principles of this mathematical philosophy.

§ 10

The Idea or Logos is in fact the ultimate unit and basis of thought. And from its nature and contents all things are evolved in the relationship of subject and object.

The union of subject and object in the Idea leads, as has been indicated, by expansion of the 'connotation' of the constituent terms to a series which is not in every case infinite, but either converges as it were into the (or an) Absolute Idea; or else returns first into itself in an ever-recurring series as in a repeating decimal: a very appropriate illustration (a converging infinite series with a definite value); and the logical propositions likewise merge into the Absolute.

This property of some ideas, and not of all ideas, is of singular but supreme importance. Since in the realm of Nature we should expect some units of matter to develop out, and in time reach a stage when they would repeat the whole process with a regularity that the mere physical properties of atoms or electrons would not have led us to anticipate. The appeal to experience and experiment is then amply fulfilled. We have only to open our eyes and look around us to observe the phenomena of life, and perceive that matter is the prolific begetter of life upon this planet, in an ever-recurring series.

The simple tests of our theory are confirmed in other ways, more refined and more elaborate when we enter into details of the behaviour, not of organisms as a whole but of individual cells as in the primary facts of cytology. And Mendel's Law and its relation to the Law of Sex follow directly from it.

The growth, development, subdivision and reproduction of the living cell, though the result in part of countless generations of selections and survivals of suitable, and elimination of unsuitable types, giving it a tendency to develop in one direction rather than in another, a so-called *vital force*, is yet the outcome of the conditions of space-time relationships of the various parts, in virtue of certain qualities of the ideas that compose them.

As a 'Monad' the Idea carries with it the 'power and potency of all terrestrial life'. It is Matter, it is Mind, it is Life, but only in the union and relationship of subject and object, as its essential nature; and the continuity merges its opposing elements into a

harmonious grouping of its constituents within the concepts of space and time, rendering the apparently unintelligible a part and parcel of the comprehensible fabric of Nature as reality!

This is the crux of the whole matter, and the crucial test of its sufficiency. When space can gather within itself the peculiar forms which ultimately unfold in rich and varied combinations as the attributes of living matter, it is not the why or the wherefore for the combinations at any particular stage that so much concerns us, as the series which constitutes as a whole the continuous progression in time. And the association of the concept of time with such a series would entail development only in one direction, or order; whereas in space it may be in one or all of three directions simultaneously. This is in order of coexistence in space and of sequence in time; such that the former may or may not be a function of the time. But when the whole group of ideas, manifold, alters with time there is motion; and if motion involves development in the cyclic process, as described above, it would appear to entail the properties of life.

§ 11

The development of the cell depends upon the innate tendency of the subject-object idea of the nucleus on the one hand; and upon the condition or nature of the space relationships of the surrounding substance, or group of ideas, on the other.

The origin of 'variations', which the Darwinian theory of natural selection leaves untouched in hopeless desperation, becomes obvious if the starting-point in the recurring cycle is slightly displaced or modified by the union of two such organisms of ideas.

Two recurring groups of decimals when added yield a third decimal group that is likewise recurring but with a variation. Each recurring term corresponds to a smaller and smaller order of existence or scale of being.

Now as dimensions are purely relative, a change from higher to lower, or from lower to higher, would be quite immaterial, provided everything were altered in the same proportion: a third kind of Relativity. It is in fact conceivable that there is a continual and perhaps continuous transformation downwards from a higher to a lower scale of being throughout Nature. And that the apparently insurmountable difficulty of the dissipation of energy may be overcome in this manner.

The transformation from higher to lower may go on continually without end; for our ideas of space are such that no limit can be

placed to the infinitely small any more than to the infinitely large, and our nature is in such respects a middle nature. The transformation from one scale of being to another is a matter of which we should be quite unaware. Just as there is progression in direction through space so can there be a continuous change of dimension. M. Bergson's creative evolution as a section of reality in time applies not merely to our familiar notions of a world of constant dimensions. If the essence of reality is change, it could be change not merely as Bergson supposes, a recreation in time, but likewise a recreation in dimension; so that, there being no absolute standard, everything may be getting bigger and bigger or smaller and smaller, without its making the slightest difference to anybody or anything in creation. Yet two important consequences follow immediately from this conception of reality. The first is that the dissipation of energy becomes a fallacious chimera due to our mistaken notion of dimension which is itself changing; whilst the dissipation of energy is but a manifestation of that change from higher to lower in the flow and distribution of energy. The second is that if all things are thus in a continuous state of change, from say higher to lower, it is extremely unlikely that they should all do so in exactly the same manner and at the same rate, and therefore some would, even if matters had already reached an approximately steady state, differ somewhat from others, and accordingly those that are slightly slower would manifest a tendency to grow relatively to the rest. This indeed would be something of the nature of a 'vital force'. Nay more, there would be a tendency to repeat the operations of one scale of being in those of another. This novel idea—for I believe it is a novel idea—opens up an entirely new aspect of the problems of life and mind as the third type of relativity theory.

Continuous variations in one scale would manifest themselves as discontinuous or abrupt mutations in the higher scale; so that Darwinian evolution in one world would be concomitant with Mendelian changes and development in the other. This is a conception that many people find it difficult to assimilate. Obviously there can be, according to our ideas of space, no limit to the transformation of scales, since the zero may be anywhere we please. The reality is in the transformation. And this is a property of space just as much as motion.

It is said that since the Universe is running down, it must once have been wound up. But this is not more absurd than the assertion of the Schoolmen that since it is in motion it must at one time have been put into motion. Motion may be as natural a condition of matter as rest, and the question might with equal force be asked,

of a body at rest, how did it dissipate or lose its motion? How was it brought to rest? This argument has really nothing whatsoever to do with Theism, though it played an important part in the Scholastic philosophy of some 700 years ago.*

The principle of relativity of dimension herein set forth shows that just as time is a transformation, ever in the one direction, so is the dissipation of energy a transformation in another 'direction', from greater to smaller. And since there is no fixed standard of dimension, this transformation may go on for ever, from the infinitely large to the infinitely small, without making any difference in the sum-total of things.

Thus the dissipation of energy which is supposed to be a gradual running down of the Universe constitutes one of the most striking manifestations of its continuous and everlasting activity!

§ 12

The concept of time as the succession of change from the future into the present, and from the present into the past, applies to every 'event', and every constituent part of the 'manifold' of an event. These are terms that will need careful and accurate definition. Every 'monad' whether consciously or unconsciously, every atom whether aware of it or not, is in time and has its own time. The theory of relativity requires that in the 'space-time continuum' every monad throughout the Universe, whatever its degree of consciousness—and consciousness does not really enter into the question—has its own time of which a conscious being would be aware if it were so placed. Such a conception appears to abolish the idea of absolute time in the Universe.

Evolution thus becomes a mere chimera, and all the elaborate calculations of astronomers, geologists, and biologists would become delusions of the untrained imagination. Darwinism and evolution generally would appear so far to be but the idle and ignorant fancies of an empty dream; for evolution implies a real past and an actual present, as it likewise presupposes a real future. Dr. A. A. Robb in his now famous *Theory of Time and Space* (Cambridge University Press) pleads for a real *now*, as well as a *before* and *after*; and in his essay on *Evolution in the Light of Modern Knowledge* (Blackie), he makes it clear that any lines of apparent evolution would be like a cinematograph film, each individual picture being true for any and every possible time according

* This is the basis of one of the five arguments given by St. Thomas Aquinas for the existence of God. It does not affect the others.

to the space elements throughout the Universe. All that is constant or invariant is the relationship or interval

$$dx_1^2+dx_2^2+dx_3^2+dt^2, \qquad (1.00)$$

in the four-dimensional 'space-time continuum'.

The 'now' on the Earth is so far as the event of the visibility of Sirius is concerned forty years behind its occurrence at that star; and likewise sixty years as regards Capella; whilst recent observations show that some of the 'spiral nebulae' are no less than 600,000 (light-years) from the Earth.

Dr. Robb shows that it is possible to arrive at a description of events in which there is an absolute time, a universal 'now' throughout the Universe.

Although the theory of relativity as developed by Einstein does imply this indefinite range of possible time, it does not require it as a necessity. And if it could be proved that there is a universal 'now', it would not dispose of the theory of relativity, but merely restrict the boundaries of space, as well as those of time, confining them to limits well within their otherwise possible extent.

If, however, all 'monads' in the Universe are, *approximately* at least, beating time (not in actual harmony, as Leibnitz supposed), time would be neither absolute, as Robb requires, nor arbitrary, as Einstein's theory implies.

If there is a slight difference of 'phase', so to speak; if some 'monads' are slightly in advance and others lag behind, an 'average time' would take the place of universal time.

If time for some atoms is somewhat faster than for others, matter would possess the qualities we propose to ascribe to it; qualities which in fact experience proves it to possess.

Such variations in time, whether periodic or otherwise, about the average and approximately absolute time would be accompanied, according to the theory of relativity, by variations of dimensions.

The slight variations which thus make an approximately absolute time possible are due to the fact that the differences in velocities between bodies in the Universe is comparatively small, relatively to the velocity of light. The Universe is not moving quite so fast after all, so that evolution and a common history are once more rendered possible.

Some spiral nebulae have recently been found to be moving at a velocity of 1,000 kilometres per second away from the solar system. But this is not so very much relatively to the velocity of light, 300,000 kilometres per second.

Bodies that are ahead in time, though not so in space, and

those that lag behind, will have the space between them apparently abnormally curved, and thus tend to attract each other. Those that are behind will repel each other as also those that are ahead.

The time in an electron moving with a velocity approximately equal to that of light will be abnormally altered and it will repel others of a like nature, but attract protons and ordinary matter whose time is behind them.

Apart from its motion, however, the electron may be nothing more than matter that is ahead or behind the proton in time. This would create an apparent strain between them in space. Hence the curvature of space between attracting or repelling bodies is in fact a consequence of their differences in time.

In reality the supposed curvature of space may be dispensed with altogether in the theory of relativity. All that is needed is that there should be differences in time, some greater, some less; but no pre-established harmony such as Leibnitz required, in which all bodies are supposed to beat time together as so many musicians in the great orchestra of the Universe, in a universal now. Nay, rather there is only an approximation to harmony, and a fair amount of discord. Some are ahead and others lag behind, at any rate in that part of the Universe with which we are acquainted. When the time between two monads is different and changing the space between them is apparently curved.

In molecular dimensions such effects become more marked if the relative velocities are very great. The dance between atoms as in a living cell is equivalent to variations in their differences of time: so that the rhythm of time is accompanied by the marvellous interchange of atoms in the living cell like a dance of Lancers or a set of Quadrilles in the regularity and rhythm of their movements, their interchanges of partners, and the almost purposeful play of their convolutions; their bewildering caresses and caprices, and the maze from which they gradually develop into the organism of living matter.

The rhythm of music becomes as it were the rhythm of life itself, and the laws of Nature the counterparts of harmony in the world of Art. And the mysterious groupings of the biological units (or determinants as they are called) which we witness *en masse* with almost hopeless awe through the microscope, in watching the subdivisions of the chromosomes in the living cell, all tell, in visual language, of the rhythmic processes in the variations of time within the monads that compose them.

This indeed is no idle dream, but the logical outcome of the operations of the principle of relativity, between the constituents

of living matter. But how or why is this variation in time within matter maintained? That itself is a property of ideas. Life is a tune in the majestic concert of Creation, and every man and every woman and every living thing plays its own part in the music of the whole. At first sight, indeed, this may seem to be mere poetry; but as we shall see it is as much the science of life as it is the poetry, and still more so the very logic of science. And as the veil of Isis is gradually lifted from her eyes, so is the vision of reality rendered more perceptible, till at length as of old we seem to see her face to face, eye to eye, and to be in the presence of the Goddess of Goodness, of Beauty, and of Truth revealing to us that which IS, HAS BEEN, and WILL FOR EVER BE.

To be in tune with the Infinite, if only for a moment! that is the life eternal! Imperishable Truth, of Identity and Diversity; of Unity and Plurality, which ever and anon elevates the mind of each of us to the knowledge of the Universal; the Self-Existent; does it not uplift the soul to a perception of the Eternal? Within, behind it all, is that Divine Light, the Inner Spirit, that Heavenly Power which moulds Creation's being, to 'that far-off Divine event towards which the whole Creation tends': unfolding Nature and Man by intellectual synthony and spiritual harmony, till eventually he finds himself in tune with the infinities of the Infinite. Even as Milton addressed his Celestial Muse for inspiration, so might it be permissible again to invoke the self-same Spirit in humble but fervent supplication:

> Thou who didst inspire that Shepherd
> Who first taught the chosen seed,
> In the beginning how the heav'ns
> And earth rose out of Chaos;
> Instruct me . . . What in me is dark,
> Illumine; what is low, raise and support;
> That to the height of this great argument,
> I may assert Eternal Providence,
> And justify the ways of God to Men.

CHAPTER I

THE IDENTITY OF OPPOSITES

§ 1

IN much philosophic as in scientific investigation the same fallacies keep cropping up with the same persistency, as though they had never been met with before in the whole history of thought. Whereas if most modern writers would but acquaint themselves with what others had already thought and done, they would save each other much unnecessary trouble, and their disciples the pangs of affliction into which their misguidance had led them, with the trouble of retracing their steps from the quagmires and pitfalls that the experienced guide would have known more skilfully to avoid.

The materialism of Haeckel is a case in point. This indeed is all the more striking as a mere prejudiced reaction to the idealism of the preceding generation in the country where the critical philosophy had had its birth. One might in fact readily imagine that Haeckel had lived before Kant, and that Fichte, Hegel, and Lotze had never been heard of. That in truth he was perhaps the intimate friend of Diderot and Voltaire; so indifferent is he to the work of the greatest thinkers in the realm of philosophic speculation of the period immediately preceding his own. The Antinomies of the Pure Reason are still disregarded, and philosophers and scientists continue to range themselves on one side or the other, as though the limitations of the human understanding in dealing with such problems had not been previously discussed and analysed; as though perhaps the key to the solution had not been found in the transcendental ideas by the exhaustive and destructive criticism of Immanuel Kant. It is true that his views are not acceptable in their entirety to those who have considered them most fully; and his followers have had to break new ground instead of actually pursuing the precise paths which he traversed. But to cover once more, approximately at least, the ground that he trod; to explore the regions he discovered, and described with such unparalleled exactness; to salute him also from afar, with all due reverence and respect, is the least we might expect from any conscientious thinker and earnest aspirant to fame in that domain of scientific speculation and serious philosophic thought.

Scientific researches, if consistently pursued, lead inevitably, whatever apologists might say, to results that, so far as the Aris-

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totelian logic is concerned, must be regarded as ultimately inconsistent dogma. They lead by rigorous reasoning to strictly inconsistent views of life and the world around us if we start from the opposite ends of thought, accepting the most extreme principles as fundamental premisses. The scientist finds himself in fact very much where Kant did, except that his dilemmas are not confined to the four antinomies, as Kant imagined himself to have been limited; but in truth to contradictions that occur, wherever he might turn his eye, and whatever he might contemplate. Everywhere and with everything he experiences precisely the same effect, the world and all that is in it two-sided, and consistent only in some transcendental sense.

The anatomist, like Sir Arthur Keith, who adduces the most incontrovertible evidence of Man's descent from lower types of animals, does not emphasize this point. The mere difference in degree between the intelligence of man and ape, the similarity in anatomical structure in every respect is essential to the point at issue; hormons being sufficient it is said to have in time brought about the almost insuperable difference between self-conscious man and his mere mimicking apish relative. The assertion is not merely that man is as a matter of degree an ape, but that in the same degree reversed an ape is really a man; this assertion only serves to intensify the differences between them. For whatever Darwin may have thought, or whatever weight might still be attached to his views by the psychologists of to-day, even if we include the opinions of the metaphysicians whose proper sphere the subject justly encroaches, it will never be nihil. The difference is not one of degree but essentially of quality; unless indeed we regard continuity as the ultimate connecting link in discontinuity, as indicated above, for the immense difference is hidden in the minute structure. Intelligence may be a matter of glands, hormons, and convolutions. But the secret of the difference, whatever that secret may be, is enshrouded in an impenetrable mystery, as inscrutable to the anatomist who views mind as an outsider, as it is to the philosopher who views it also internally and views it whole. He views reality as the link between mind and matter; and mind as the link between matter and reality; not merely matter as the link between mind and reality, which it undoubtedly is, though something more, far more. Without mind neither matter nor reality can have meaning, and as I venture to show still less be. The full-grown man does not lose in dignity or importance from having developed as an individual from a germ of illimitable ancestry. Why should racial (phylo-

genic) as distinct from individual (ontogenetic) development affect his spiritual and intellectual status? To be related to apes of more than 20,000 generations ago is surely less disconcerting than to be immediately related to most of those who still believe the World to be flat, the centre of the Universe, and man to have been not more than a few thousand years on this Earth. In other words, to be so nearly related to those as far remote in knowledge and intelligence as the gorilla or the chimpanzee, or, as far as we are concerned, a gourd; for the difference between a pumpkin and a bumpkin may be not much more than Grimm's 3rd Law requires, since in the process of time *p*'s are gradually transformed into *b*'s and all without the aid of Mendel!

Development in time is a conventionality, like the presence at the same time of different individuals in varying degrees in space. We may take any category whatever, not merely one of the ten Aristotelian or twelve Kantian categories, but any categories whatever as Hegel would have done; and predicate any predicative class of another that is predicable by it, and we shall find that the same thing holds, that the development along any coordinate can be conceived in continuously varying degrees where time is a constant for any particular event. And when time also varies, the result is said to be a development in time. The difference is like that which a Martian would experience finding himself on this Earth when observing the various ages of men he encountered, unconscious of their growth as contrasted with the effect on one of us who had witnessed the growth of these in time.

Among the four variables we may regard any one as constant, and we should have sections of reality, three being projections in space, in the *yz*, *zx*, and *xy*, planes in a varying time; whilst the fourth would be the event at a particular moment; in the three dimensions of space. The four-dimensional continuum constitutes the sum-total of 'space-time' existence. In any case the result is the same as though it were viewed from the standpoint of some particular individual. But the relationship of each individual to every other is itself an event, postulated no doubt, and yet without any rational meaning unless an approximately universal time be postulated, such as the theory of relativity endeavours to dispense with. Yet an event in common amongst all monads is from a physical point of view not only unessential, but difficult to reconcile with the relativity theory. From the metaphysical point of view such a relationship may seem necessary, though Physics has no need of it. There is, however, an approximate resemblance between the various monads. And apart from the

need or presence of a universal time, we may assume an average time round which the time of monads would range; and in this manner history and evolution are rendered possible. This view of the question with its consequences must be fully considered. The ranging of time within certain limits according to the law of probability, like the law for velocities in the kinetic theory of gases, is the natural outcome of this view. For although the monads are not all in tune nor strictly speaking can any two actually be so, yet the Infinite Monad should be regarded as giving what might be considered the 'universal time'. This is what strictly speaking is. Leibnitz held that all monads must be in harmony with this One; but it is by no means essential that this should be so. On the contrary it appears necessary that it should not be so; since only then could the irreconcilable become reconcilable, and the conflicting dualisms and multeities united. The existence of Truth itself must convince even the greatest sceptic who believes the sense of his own assertions that a Universal Monad and standard of reference is absolutely necessary, if a discussion between more than two monads at a time is to be considered possible; for not only the phenomena of space and time are involved, as the pure physicist is prone to imagine, but all operations of the understanding are generally performed through the categories or windows of the mind.

The essence of reality 'das Wesen' was arrived at by perceiving that the contradictory is that which gives ballast and symmetry to the whole. To learn to distinguish the 'same' from the 'different' and identify the 'different' with the 'same' in certain but different circumstances—this is the task of higher Hegelian logic. Obviously the theory of relativity offers an excellent example of so great an anomaly. In the absence of a Universal Monad, however, truth is impossible. An approximate truth like an approximately universal time, the average of all possible times, will not do; for an absolute standard of truth is indispensable as a basis of certainty; else all thinking would be futile, illusory, imaginary, with no definite standard of value. Life would be but an incoherent dream, satisfactory only to those who do not think things out to the full logical conclusion; with a philosophy of 'act and make merry, for tomorrow we die' as a comforting theme; an intelligence of the fox-terrier type perhaps; cunning, skilful, alert but not thoughtful, reflective, or thorough, if even rationative. In Kant's terminology, understanding is distinct from reason; to which Science thus gives the synthesis, or the possibility of one. Its efforts appear one-sided, standing in mid-air, like a proposition or sentence lacking a sub-

ject, a verb, or an object. The flight of pure reason may be said to extend to two worlds of existence; to unite two continents of reality, for which the science of the nineteenth century offered no clue, no shadow of hope or consolation; but for which transcendentalism presents an earnest and practical solution. No transoceanic flight can equal the glory of so great a feat if it were so accomplished, to be able intellectually to stand erect, as it were, and extend the right hand to Plato, and with the same nisus give the left to the men of this Earth.

We now perceive in a new light, perhaps, the same old problem which puzzled the ancients from the time of the Ionic school, 600 B.C. onwards. The problem forms the fundamental thesis of philosophic discussion; namely, 'What is the nature of that unity which lies behind the multiplicity and difference of perceived objects?' Whether we regard that unity as the infinite with the Ionians, or as God Himself with their successors the Eleatics, we would appear to be merely repeating history. But not really so. For on analysis the unity of the one and the many is a mathematical necessity, from the laws of thought. The unity and plurality are separate categories, which happen both of them to relate to quantity; but it is only like the unity of other superposed or simultaneously contemplated categories or the concepts of space and time.

The combined effect is like stereoscopic vision, due to seeing the same object from two different points of view at the same time. An illusion is produced which makes it appear that the two different views are really one; and at the same time gives the impression that bodies do stand out in space; whereas it is merely the overlapping of approximately alike pictures. The resultant effect is a unity of diversity akin to the continuity in space or time. There is an overlapping of consecutive impressions or events. The imaginary roots of unity, involving $\sqrt{-1}$, serve to bridge the gap in algebraic logic. That which results from two different propositions held simultaneously as one and the same is something really different, and virtually amounts to an 'emergence'. The synthesis from opposites in the Hegelian dialectic. Only the recognition of the part played, and that a most important part, by the imaginary roots of unity in the equation of identity will enable the gap that exists in reality to be filled up. It must be remembered that the doctrine of geometrical continuity makes the apparent discontinuity appear to be but a particular case of continuity extended to infinity. A line which returns into itself, by way of infinity; parallel lines which intersect thereat; asymptotes that touch their curves; spheres which

intersect in the same circle at infinity; and so on; and all circles that pass through the two fixed circular points at infinity; the tangents from these two points to any conic intersecting in two points, none other than the foci of the conic; all these propositions sound like romances of the mathematical world. The unity of the one in the many is nevertheless not in any way explained by these excellent meditations. For the problem involves the question of the actual overlapping of ideas as well as classes; this is precisely the same old problem over and over again, merely in different words. But when two classes apparently distinct do not overlap in any real members, they do nevertheless necessarily overlap in at least two members at infinity. And that which is absolutely true at infinity does possess the characteristic of continuity in the finite world around us. This is not a mere matter of analogy. Rather is it a matter of 'necessity', in the very nature of thought; whether it be viewed algebraically or geometrically as a logical accessory to the process of cognition. Yet throughout the whole realm of Nature and of thought, we are met continually by the continuity of the apparently discontinuous; or if we should prefer to put it so, for it ultimately comes to the same thing, although it may apparently mean the exact opposite, the discontinuity of the apparently continuous. The identity is really a slowly changing difference. It is in fact an extension of the Greek 'method of exhaustion' due to Eudoxus, and followed by Euclid and by Archimedes, which led to Newton's theory of fluxions. It was Archimedes who, in finding the area of a parabolic segment, and that of a spiral, the surface and the volume of a sphere, or any other solids of revolution, is said to have extended the 'method of exhaustion' or integration of infinitesimals that later formed the basis of the integral calculus. The development of this idea was revived at the beginning of the eighteenth century when a curve came to be regarded as an assemblage or sequence of points; or as a sequence of infinitesimal straight lines; an area as a collection of infinitesimally small squares, or sequence of infinitesimally narrow strips; and a volume such a collection of infinitesimally small cubes, or sequence of infinitesimally thin laminae; a tangent again as an extension of an infinitesimal chord. And by the introduction of the idea of time, in conjunction with the three-dimensional space, we obtain the notion of an 'event', when the element of time is infinitesimally small. The summation of events in time constitutes the space-time continuum. And if the element of space as well as that of time be infinitesimally small, then we may form the conception of an 'interval' which forms the basis of Einstein's general theory of relativity. Strictly speaking, the

infinitesimal is now abandoned and we deal with points or events. When in a similar manner time is taken with energy, or what is now regarded as the same thing mass, then we obtain the notion of 'action'. This conception of 'action' is perhaps the most important in the whole realm of modern physics. But during the last twenty years or so quite a revolution has been effected by the work of Weierstrass, Dedekind, and Cantor. The infinitesimal has vanished and we have to deal with actual points. But continuity remains.

We owe to Lord Russell and Dr. A. N. Whitehead the first real step in advance in this direction; and they have achieved it by the aid of symbolic logic. While pursuing an entirely different path, I have endeavoured to bear in mind their great achievements in the clearing up of our ideas on fundamental points. Their theory of Types is of supreme importance in avoiding many familiar fallacies. They have confined themselves to the mathematical rather than the purely metaphysical difficulties of the subject. While expounding with admirable lucidity and precision the theory of the infinite, they do not appear to have considered sufficiently the question of identity. And the real problem of metaphysics, as Mr. F. H. Bradley insisted, is the question of the unity of diversity in identity. Hegel is not yet dead. And my object has been to show that the symbolic logic of the earlier kind invented by Boole leads to the solution of problems not easily to be tackled by other means.

I have sometimes felt that some of my contemporaries might allege I had written this book in a balloon, so isolated is my endeavour to do justice to the many who to-day are overlooked, particularly the two great founders of symbolic logic, Gottfried Wilhelm Leibnitz, George Boole, and Hegel the founder of Synthetic Logic. This book was not written in a balloon but it was written in Merano, Italy, where I have resided for the greater part of the past six years. A voluminous library including works like *Principia Mathematica* is difficult to carry about, the latter perhaps more difficult to read and indeed to understand, nevertheless I have endeavoured to keep in touch with the latest developments of the subject while pursuing my own independent course and to some extent, I think, original path, without encroaching blindly upon the researches of others. While following with admiration the magnificent work of Lord Russell on the one hand, and of Dr. Whitehead on the other, and of both when they combined, I still venture to think that Bradley has been ignored or else misinterpreted and misunderstood, and that the

Hegelian school still merits more attention than it has hitherto received from scientifically-minded philosophers in our time.

To the notion consisting of the particular and the universal; form and matter; or substance and accidents; differences in identities as species and genus; in the veritable *in re* of Aristotle's later metaphysics Hegel adds the individual. With Leibnitz, on the other hand, the principle of individuation is the monad, as the substance and the form. But the matter is the resultant of all the various qualities superposed, as perceived by other monads. It constitutes the accidents, and has no reality apart from the recipient monads by whom they are perceived; unless they are, as ideas, permanently stored as 'eternal things' in the great storehouse of the universal, as universals and permanent possibilities of perception. The 'this-ness' of a thing is not expressed by its accidents or material qualities, physical or chemical; for these are its accidents, as previously insisted upon. They are universals 'illuminated' by time, which time draws out of the limbo of the noumenal reality itself not necessarily in time, but which constitute the basis of all thought as the transcendental in pure reason. The unity of thought in the individual is then possible only because reality in the universal monad exists as the permanent basis of truth. Without this all is meaningless and chaotic. Truth is the harmony of ideas. And the relations in space and time are particular types of such harmony. The unity in identity as expressed by the relation $x = x^n$ shows that the 'this-ness' is not as simple an affair as we might at first have imagined. x^n is not x except when x is 1; that is when x is reality. The equation, however, is satisfied by the imaginary algebraic roots of unity (*vide* pp. 196, 199). The unity of all things in the Absolute when regarded as particular in the Highest Universal, this is the Great Reality before Whom every knee must bend and every head must bow, and acknowledge the ignorance and limitations of men; priest, philosopher, one and all; and Tyndall too.

No idle empty rhetoric is expressed by these words. On the contrary the 'form' in the particular individual, its substance or 'this-ness', is identical with the universal when it is the subject of the particular union through communion of spirit with the Absolute Being. The Platonic universal is merged in the 'form', as the form of the universal and the particular become one, in the predicate and subject of a proposition, one as a logical unity and identity.* To take a simple illustration.

* Russell and Whitehead's theory of types does not surmount the difficulty.

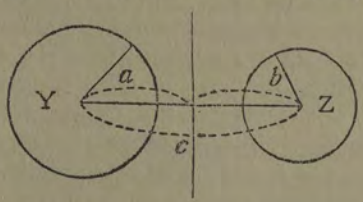
If X is Y, and X is Z, but Y is not Z; then although a contradiction no doubt ensues as regards all finite qualities, there nevertheless remains a common element that might be regarded as the 'form' in the unity of X, Y, Z; when Y and Z are classes of different types and do not overlap.

For instance, the two circles Y and Z do intersect in two imaginary points.

For if their equations be $x^2+y^2=a^2$ and $(x-c)^2+y^2=b^2$ the coordinates of the points of intersection are:

$$X_1=\frac{c}{2}+\frac{a^2-b^2}{2c}\text{; and } Y_1=\frac{\sqrt{-1}}{2c}\cdot(a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2).$$

It can easily be shown that the quantity in brackets is always positive when $a+b<c$; so that Y_1 is imaginary. It should be noted, however, that whilst the y coordinate is imaginary, the x coordinate is real. The former line being $n=\frac{c}{2}+\frac{a^2-b^2}{2c}$ represents the 'radical axis' of the two circles, i.e. the locus of all points, the tangents from which to the two circles are equal; or again the locus of the centres of circles that cut the two given circles ortho-



gonally. This is a straight line at right angles to the line joining the centres of the two given circles, and dividing it in the ratio

$$\left(\frac{c}{2}+\frac{a^2-b^2}{2c}\right):\left(\frac{c}{2}-\frac{a^2-b^2}{2c}\right).$$

This radical axis is the common chord of the two circles; and it is a real line. Of course this line is a circle whose centre is at infinity. Hence, if the two classes Y and Z are represented by two circles the class $\bar{Y}\bar{Z}$, i.e. not $-Y$ and not $-Z$, must be regarded as divided into two sub-classes one of which includes Y, and the other Z; and there exists a class within $\bar{Y}\bar{Z}$ some of whose members are common to Y and Z. When then one says that Y is Z, although Y and Z are two different types, one means this, or one means

The proportional functions $\phi(xy)$ and $\psi(xy)$ do intersect in n points though imaginary ones: yet they may be represented algebraically.

nothing at all. When the subject S of a proposition and its predicate P, which are different ideas, are identified by the proposition S is P, whether $S \supset P$ or $S \in P$, i.e. all S is P or S is a P, we mean this and nothing else. The uniting principle then, if subject and predicate be regarded as distinct, can only be conceived as transcendental truth, of the algebraic nature, with perfect continuity of thought.

It is not right then to say with Plato that the predicate is real as the universal, but that the subject is merely a phantom. Nor with Aristotle, as he is popularly supposed to have held, that the subject is the real substance, and the predicate non-existent without it. In a proposition the two co-exist and are inseparable so far as that which is in common between them is concerned. If S's are not P's, and P's are not S's, then S and P are independent of each other. But there is an SP dependent on both. For SP is a reality as much as either S or P separately when referred to both united. It is not merely the Platonic universal P that is real, but the subject S, and no less so the proposition embodied by S is P or SP. Aristotle did not admit that the universal P existed independently of the subject or particular S. The predicate P is a real universal; but only in the individual subject. Both in the *Categories* and in the *Metaphysics* Aristotle maintains as his fundamental position that 'all things are substances with attributes as universals'; but that these latter have no separate existence (contrary to what Plato taught). At the same time there is a curious tendency on the part of the pupil to revert to the master's teaching; for he regards the substance of the predicate as other than an individual substance, for such a primary substance is not a predicate; but the species or genus is the substance which is the predicate of the subject. Thus 'Socrates is a man and an animal'. Here the substance of Socrates the individual is not that of man, the species; or of animal, the genus. Substances are then real predicates; and as they must be either species or genus, it follows that some substances are species or genera. These were called 'secondary substances' by Aristotle, and by the Neo-Aristotelians *substantiae universales*. The universal species 'man' or universal genus 'animal' is itself a substance; though contrary to what Plato taught, never existent except in the individual man or animal. This is the earlier teaching of Aristotle as given in his *Categories*. But in his *Metaphysics* he endeavours to shake off the influence of his master Plato altogether, an attempt which he never succeeded in accomplishing; for although he denies in the *Metaphysics* that universals are substances at all, he introduces the idea of a universal *essence* of a

species of substances which he calls substance (*οὐσία*); for 'the individual essence of an individual substance really is that substance, and the universal essence of the whole species is supposed to be indivisible and therefore identical with the individual essence of any individual of the species.' (T. Case, 'Aristotle', *Encyc. Brit.*, 11th edition.)

'Since a separate individual, e.g. Socrates, is a substance, and he is essentially a rational animal, then his *essence*, being what he is, is a *substance*; for we cannot affirm that Socrates is a substance and then deny that this rational animal is a substance. Now according to the unity of a universal asserted by Plato and accepted by Aristotle, the universal essence of species, being one and the same for all individuals of the kind, is the same as the essence of each individual: e.g. the rational animal in the human species and in Socrates is one and the same; 'for the essence is indivisible' (*ἄτομον γὰρ τὸ εἶδος*). It follows that we must call this self-same essence, at once individual and universal, substance—a conclusion, however, which Aristotle himself never drew in so many words, though he continued always to call essence substance, and definition a knowledge of substance.' (T. Case, loc. cit.)

The striking inconsistency as it appears to me in the Aristotelian philosophy is that the substance or essence of a thing is a predicate and therefore a universal; and if it has an independent existence as substance which its other qualities have not, it is at any rate a Platonic universal. And if this universal is possible, why should other Platonic universals not be so?

Berkeley, of course, showed the absurdity of this incoherency. For what is 'sauce for the goose is sauce for the gander'. The universal in one case as substance or essence is not more real or independent than any other predicate in the Platonic sense. Hence, either neither one nor the other universal has any real independent existence or they both have it. Berkeley maintained that neither had an existence independently of perception. But the Idealistic-Realist of to-day, whilst admitting that only ideas exist, believes in the external world, as a world of ideas, existing as the Platonic universals in the Divine Mind, or else as some permanent 'mind-stuff'. Every idea can exist as subject or predicate. And it must exist as one or the other, or both. All combinations of ideas are possible, in the abstract, but only those that emerge in the development of the series in time (p. 244) actually exist, or are possible to our consciousness, in so far as they must conform to the categories of the understanding, in the Kantian sense. Memory is only of ideas that have been actually experienced. But all possible ideas are akin to those of memory, although they might not have been actually experienced; and as such are potential experience,

this virtually being equivalent to the Platonic theory that cognition or perception is nothing more than memory. But it differs from Plato in not requiring that the memory be our own, for we may never have experienced it, being memory in the Divine Mind, eternal, immutable, ever-permanent and ever-true; the essence of all substance, and the source of all that is, has been, and ever is to be. This is the God of Aristotle, but far more; for He is likewise the God of Plato, and still far more. He is in truth the God of Hegel, and yet still more. For He is the God of all gods, Whom none of us can conceive save in a partial light; the Infinite of all infinites; the One Universal Monad; the Absolute and Perfect Being in Whom all things are One, and thought becomes possible in that unity which constitutes diversity, without which there would be absolute discontinuity of types. Events would be a series of jerks. It is this Mind that supplies the continuity.

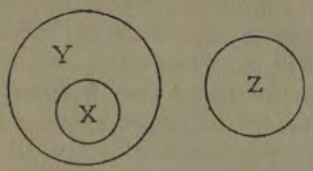
In the equations $x = x^m$, and $y = y^n$: then $x = y$; if $x^m = y^n$; and this latter must be the case if m and n are equal to unity or infinite sufficiently large. Just as parallel lines intersect or as all circles pass through the two common points I and J at infinity, Non-Euclidean geometry regards this as no mere fiction of the imagination but as actually true.

Moreover, all adjacent points in geometry may for practical purposes be regarded as coinciding, though distinct. This corresponds to $x = x^2$ in logic as we shall see later in the case of the point of contact of a tangent to a curve, or of two curves with each other. Again, three points may coincide, as when a tangent not only touches but actually cuts a curve: i.e. in $x = x^3$. And so on, with nodes and cusps, stationary and double tangents, or inflexions, all of which relate to what are called singular points, in the higher plane curves. We may have $x^m = ky^n$, an interesting case of the higher parabolae with the characteristics under consideration. All curves, however $F(x, y) = 0$ and $f(x, y) = 0$, must intersect each other, if not in real then in imaginary points at infinity. All propositional functions do overlap algebraically if only in imaginary values of the arguments or independent variables. The precise number and nature of these are given by Plücker's well-known six equations connecting the number of double points, cusps, double tangents, and inflexions in any curve with its order and class; the *order* of a curve being the number of points in which it is cut by a straight line, and its *class*; the number of tangents (real or imaginary) that can be drawn to it from any point in the same plane. Now in logic we are not as a rule concerned with the actual shape of a curve that may be drawn to represent the

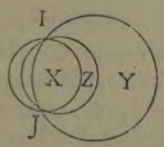
class of objects supposed to lie within a certain boundary; or to be represented upon the curve itself. But just as a curve can represent an equation, or an equation represent a curve; so either or both can be made to stand for a logical proposition; and the points of intersection must represent that which the various classes so represented have in common. If the classes are mutually exclusive, as in the proposition 'No X is Y'; then X and Y may be regarded as having something in common; not necessarily at infinity, as in the case of geometry, but possibly in the noumenal world; though certainly in some sense, howsoever indistinct yet 'als ob'; 'as though', it were so, $X$and Y do overlap, and the contradictory 'Some X is Y' is true. The two classes X and Y do intersect if not in the world of experience or phenomena, then in the world beyond experience, namely that of noumena. This approximates to Kant's transcendental Idea in mathematical form.

If in the world of experience, however, we have 'X is Y', and 'X is not Y', this means that X and Y have at least something in common and either merely touch or actually touch and cut each other having one or more double tangents in common.

The Hegelian contraries do come under this conception; for 'All X is Y', and 'No X is Y', are still not irreconcilable and can be held simultaneously, as consistent at infinity. Here we must fall back upon the nature of identity as in $x = x^n$; and its bearing upon the law of contradiction; or else since 'All X is Y' may be represented by a circle X within another Y and 'No X is Z' a circle X completely outside another Z, then all three XYZ intersect in the two points I and J at infinity:



Here and Now.



At Infinity.

§ 2

The *Wissenschaft der Logik* astounded the intellectual world with the bold analysis it contained of the nature of Identity, and the still bolder identification of the idea with the object. After a fierce storm of bitter controversy, of vulgar vituperation, and when these failed ridicule of a still more ignorant and despicable sort, the philosophic minds of Germany were converted to Hegel's views.

His following and the influence it exerted, for a time at least, probably exceeded the authority of Kant in a previous generation. But it was not destined, as Kant's was, to be so lasting, at any rate amongst his own countrymen; and the reaction that set in subsequently not long after his death might have extinguished his influence for ever had it not been that, as Matthew Arnold has so well said, 'good German philosophies when they die go to Oxford'. In 'the home of lost causes and forsaken beliefs, of unpopular names and impossible loyalties', it found a new home, and might be said to have changed its nationality; for the best Hegelians of the past generation were Englishmen. William Wallace and the two Cairds gave a fresh stimulus to the revival of learning in its deepest sense.* The stimulus penetrated to Cambridge where, as history relates, the innovation was at first regarded with horror, as a faint echo of the superstitions of the Oxford Movement, or some other imbecility; though later more mirthfully treated with contempt, or scornfully referred to as mere 'Oxford rot' and 'German rubble' transported from Greece in the denudation of time. In Cambridge, however, it found after a while an able expositor and original critic and thinker in the late and ever to be lamented Dr. J. E. McTaggart; and in Dublin it was expounded at Trinity College by the late Professor Thomas Maguire whose published lectures served to guide, even in a later generation, the reading of the more thoughtful within its walls. His successor, Professor H. S. Macran, has also added to our knowledge by a valuable study of *Hegel's Doctrine of Formal Logic*.

So far as I know the whole of Hegel's *Wissenschaft der Logik* has never been translated into English. Only the first part appears in Hutchinson Stirling's *Secret of Hegel* and the second and third in Macran. A translation of the 'Logic' in his *Encyclopaedia* has been given by William Wallace, together with a valuable Prolegomena of his own, as an introductory volume.

The statement *A* is *B* must imply that some *B* is *A* unless *A* and *B* are identical, which in fact they are not except as regards the particular *B*'s that are *A*'s. *A* and *B* are identically the same so far as the relation is concerned, but *B* is not *A* in any other respect; that is to say, that every *B* that is not identical with *A* is not *A* at all; in other words, *A* is not *B* except when *A* and *B* are identically the same. In truth the *subject* is *not* the *object*, yet in a proposition they are identical, because so far as the proposition goes they are identically the same.

* In Italy it is well represented by Croce and Gentile as Neo-Hegelians.

In another sense the paradox of the Hegelian compatibility of contraries and contradictories is but the outcome of the principle of continuity, with which we shall deal later.*

The statements A is B_1, and A is not B_2, may of course both be true. Now if B_1 and B_2 approximate, they will in the limit become identical as B_3, in which case A is B and A is not B will both be true, though still not true in the Hegelian sense. For instance, in passing from a position within a circle to a position outside it a point is passed that is neither inside nor outside it and yet both inside and outside. It is in fact in the transition stage between B_1 and B_2. And yet it is on the circle.

The non-Hegelian will readily grant all this. He will admit that there are points that are both inside and outside, or neither inside nor outside; but he will add that that does not alter the fact that there are some points that are entirely inside, and others entirely outside the circle; and that it does not appear possible to contradict these facts. It might appear that the Hegelian dialectic could apply only to the transition stages between classes of ideas; when A and not $-A$ become one. A point inside a circle, and therefore not on the circumference, cannot be said to be outside the circle. But it can be said to be on the circle and yet not on it, if we consider the plane of the circle; since it may be neither above nor below it, and yet both above and below; as it is on the plane of transition both above and below the plane of the circle. If it is not in the plane of the circle it cannot be said to be inside the circle. *And so on with the n dimensions of thought.* Every statement A is B can be qualified by the statements B is C, and B is not-C; but A is B if B is C; and B is not-C; therefore A is not B. But if it be a fact that A is both B and C, so far as can be ascertained a further qualification D may render the statements B is and is not D; or C is and is not D; so that A is B; and A is not B; both become possible, since B is in the transition stage between some class BC or BD or ... BN; or CD or CE or CF ... or CN ... or CX. We may therefore have an indefinite number of A's and B's; i.e. $A_1, A_2, A_3, A_4 \ldots$

* A point is distinct from a consecutive point; but only because there are an infinite number of points between any two supposed 'consecutive' points! Thus continuity is rendered possible. The number of points in space is the same as that between any two points whatever; for there is a one-one relation between them. This constitutes Cantor's now accepted definition of the infinite. When there are two groups, one of which forms part of the other, and there is a one-one relationship between their members, they are both equal in number and infinite. The number of fractions between 0 and 1 is equal to the number of cardinal numbers altogether, for there is a one-one relationship between all possible numerators and denominators and their combinations. (See B. Franckes, loc. cit., &c.)

A_n, and $B_1, B_2, B_3, B_4 \ldots B_n$. There is always some qualification which would render the proposition A is B not absolutely true, so that A is not B is true, and yet not absolutely true. The absolute alternative applies only to the Absolute which cannot be contradicted. Then and then only is there absolute identity; when $A \equiv B$ is strictly true.

The Hegelian dialectic thus implies a continuity and gradation in all ideas and relationships. It is assumed that this exists between all ideas, even between opposites, as a necessary result of the infinity of attributes; by one of which an idea can ultimately be qualified; unless the Absolute itself is contemplated, since it cannot be contradicted and be absolute. Likewise in the world of reality the same contradictions are found to exist. As J. H. Muirhead remarks, in 'Idealism', v. *Encyclopædia Britannica* (1911), 'Ideas when held in their *abstraction* are *almost* interchangeable with their opposites —extremes meet and in every true and concrete idea there is therefore a coincidence of opposites.' And again: 'The doctrine of the unity of contraries, so far from being the denial of the law of non-contradiction, is founded on an absolute reliance upon it. Freed from paradox, it means that in every object of thought there are different aspects or elements each of which if brought separately into consciousness may be so emphasized as to appear to contradict another. *Unity* may be made to contradict *diversity*, *permanence change*, the *particular* the *universal*, *individuality relatedness*.' The problem of the one and the many presents itself with renewed force and vigour. It is a problem that no rational thinker can afford to ignore or refuse to consider.*

'Man is man,' says H. W. Blunt, 'and good is good; but to say that man is good is already to say the thing that is not. Plato replies with the doctrine of the interpenetration of ideas, obviously not of all with all, but of some with some, the formula of identity in difference within thought itself. . . . The proposition and judgement involve subject and predicate. . . . Plato's Division or the articulation of a relatively indeterminate and generic concept into *Species* and *Sub-species* with resultant determinate judgements, presumes of course the doctrine of the interpenetration of ideas laid down in the Sophists as the basis of predication. . . .

'The ideal is to determine progressively a universe of discourse till true *infima species* are reached, when no further distinction in the determinate many is possible, *though there is still the numerical of the indefinite plurality of particulars*. The process is to take as far as possible the form of a con-

* Russell recognizes that 'events' like points are distinct, but supposes a relationship between them, if I understand him rightly.

This relationship is itself an idea and it is of a different type from the ideas it relates. In the limit, however, there is no relation other than that of continuity which constitutes identity in difference. It is here, I think, that Bradley makes good his point.

tinuous disjunction of contraries. We must bisect as far as may be, but the division is after all to be into limits, not parts. . . .

'There must be convergence in a unitary principle, soul or consciousness which is that which really functions in perception; the senses and their organs being merely its instruments. It is this unity of apperception that enables us to combine the data of more than one sense, to affirm *reality; unreality; identity; difference; unity; plurality* and so forth, as also the *good;* the *beautiful;* and their contraries.' (H. W. Blunt, 'Logic, History,' *Encyc. Brit.*, vol. xvi, p. 898.)

In the distinction between the particular and the universal we must recognize that there are different degrees of universals, higher and higher; and lower and lower till the universal merges ultimately in the particular, in the identity of both. It is not necessarily, as Aristotle supposed, a matter of induction, which claims to start with the particular and end with the universal. The continuity is Hegelian rather than Aristotelian. Berkeley did not recognize it, and like Aristotle conceded existence or reality to the individual; but according to Aristotle knowledge is of the universal; which is not real, for only the individual exists or is real: 'What is known is not real, and what is real is not known'. The Hegelian philosophy transcends or surmounts this difficulty by the doctrine of continuity in the reconciliation of opposites. Knowledge is real; and what is real is, or can be known. The gap from the lowest universal to the individual and particular cannot be considered as unbridged; there is continuity in the Absolute Universal. As Mr. Blunt lucidly points out (loc. cit.), Aristotle falls back ultimately upon the psychology of the question, to try to reconcile the two, in the principle of *νοῦς*.

The psychology of the continuity between universal and particular, and the reconciliation of opposites is best dealt with by Herbart, Kant's successor in the Chair of Philosophy at Königsberg. He might be described as the founder of the science of psycho-physics. James Ward says of him that in exactness and penetration of thought, he is quite on a level with Hume or with Kant.

The limitations of *summa genera* and *infimae species* cannot be too strongly emphasized, for these are clearly indefinable. The latter are describable only by their accidents, as the former are definable only by what comes under them. How can we expect to go beyond them?

If the species is included in the genus, as the subject is included in the object, and if the latter is an idea, how can the former be denied to be one? The fact that the subject is only an idea of a

hypothetical *thing-in-itself* does not overcome the difficulty; for if the qualities of the thing-in-itself are ideas, the relationship renders the thing-in-itself an idea too; else the relationship would have no meaning, though the qualities would refer to it. There would be no continuity; no *species* in *genera*; the alleged species would have no genera; they would have no qualities; they would be nothing; not only unknown, but unknowable, as Herbert Spencer maintained; without any possible qualities and devoid of any possible meaning; something and yet nothing; a contradiction and therefore not a reality and consequently in no sense a thing-in-itself; as an ultimate basis of phenomena; but a mere phantom of the imagination, an unknown hypothetical superstitious x, where $x = 0$ is the only solution. The synthesis is in the unity of subject and object, in which the two are essentially ideas which have merged one into the other and both into one.

The realism of Thomas Reid and Sir William Hamilton, and the pseudo-realism of Herbert Spencer have been the cause of much confusion of thought during the past two centuries, and one of the great superlative hindrances to the progress of knowledge.

'With Bradley reality is the one subject of all judgement immediate or mediate. The act of judgement "which refers an ideal content (recognized as such) to a reality beyond the act" is *the unit of logic*. Grammatical subject and predicate necessarily both fall under the rubric of the adjectival, that is within the logical idea, or ideal content asserted. This is a meaning or universal, which can have no detached or abstract self-subsistence. As found in judgement it may exhibit differences within itself, but it is not two, but one, an articulation of unity, not a fusion, which could only be a confusion, of differences.' (Blunt, loc. cit.)

This of course is the problem of continuity and discontinuity; and no thought or judgement is possible without the admission of this unity, whether it be a mergence or fusion (or confusion) of differences, as Plato supposed; or an articulation of identity and indifference as Bradley regards it.

The question is how the transition can take place from identity to differences without raising the tangle of confusion the question presents. The answer is that there is no transition.

Bosanquet, following Bradley, gives the definition of *inference* as: 'The indirect reference to reality of differences within a universal by means of the exhibition of this universal in differences directly referred to reality.' In other words the differences within the universal are pointed out, emphasized and selected, and the particular is given a place as identical with that particular difference. Thus inference becomes a substitution of identities; the true characteristic of Bradley's theory of inference.

This theory is quite new and gives Bradley a unique position amongst the original thinkers of his time. For he appears to have overcome the real difficulty in the Hegelian logic, and the confusion of the interpenetration of ideas which formed the stumbling-block in the Platonic theory; for the universal and the particular are ultimately one.

But there are no degrees of reality. Everything is either real or not real. The real is either *A* or not-*A*; but the non-real may be both *A* and not-*A*; or neither *A* nor not-*A*. And since the truth is real, there cannot be *degrees* of truth, as Hegel supposed, but only *degrees of apparent truth*, which cannot be regarded as truth at all; but merely as an illusion which does not and cannot ever approximate to the truth from which it is distinct by an absolute discontinuity, that separates the real from the unreal.

The Hegelian Dialectic postulates a continuity between the real and the apparent; so that by a series of higher and higher syntheses of unrealities between contradictories, the absolute may ultimately be attained.

This jump may appear to be an impossibility, since the real would have to partake of the unreal; and there is a perfect discontinuity between the two. Moreover, since the series might be extended both ways, by the expansion of both subject and object, the apparent and self-contradictory might be pushed to the opposite extreme of absolute negation of everything; and that would be falling back upon the unknowable.

One way out of the difficulty is to deny both the unknowable and the absolute; regarding all knowledge as self-contradictory, apparent, relative, and unreal. The other is to admit that the absolute or real though not self-contradictory is continuous with the unreal, from which it differs by an infinitesimally slight extent as in the case of points along a line. This, however, appears to be raising the same difficulty in a fresh form. The third alternative is to regard reality as relative and self-contradictory in various degrees; in the highest form of which self-contradiction is eliminated, though a perfect continuity exists between it and the lower grades of reality. This is not a play on words (as the analysis in the sequel shows, see p. 130); since the reality must be continuous with the unreality, and therefore self-contradictory; except in its very highest form as the Absolute (which if continuous with the apparent or unreal becomes self-contradictory; only when it differs from itself by an infinitesimally small extent as in the definition of the infinite and continuity, see p. 238). But then such a continuity would still seem to imply that the unreal is to some extent

real! This is best exemplified by the fact that there is continuity between the future and the present; as also between the present and the past. But although the present is real, yet the future and the past, with which it is continuous, may neither of them be real. If there is no continuity between the real and the unreal, then must time consist of a series of jumps of events, as in a cinematograph film, the continuity being in the mind that perceives.

This subject is best dealt with mathematically. It is an extension of Boole's logic to 'dynamic' thought in contradistinction to the Aristotelian 'static' thought. Continuity is a fact of experience, although at variance with the laws of thought in static logic, yet is it quite in accordance with the laws of the higher dynamic logic, which gives unity and harmony to the whole.

Bergson's theory of a continuous creation from the unreal future to the real present, and a continuous annihilation of the real present into the unreal past, implies such a continuity between the real and unreal. But the Platonic theory of ideas would regard all ideas past, present, and to come as real; and as such affords a more consistent aspect of the Whole.

If we suppose that every idea whether rational or not; whether true or untrue, is real; then discontinuity is avoided. Every idea and every proposition, whether true or untrue, is nevertheless a reality; and as such represents an aspect of reality. The future and the past become realities like the present; though they differ from it, they nevertheless partake of a continuous existence with it. The present is the real in consciousness, or reality which is being realized in consciousness. The future is the real that has not yet been realized. The past is reality that has been realized and exists as memory. Like the search-light thrown upon the background of the landscape, consciousness illumines the background of reality in the light of perception; and gives to the unrealized idea the element of an objectiveness as a time event. This is the most consistent development of the Platonic theory of ideas. It eliminates the discontinuity between the real and the unreal by making every idea real, whether it be true or not; rational or not-rational. Memory is but an after-effect, a sort of phosphorescence of consciousness. Schelling and Hegel's assumption that the real must be rational is eliminated as unnecessary and untrue. Every idea is real, but it may or may not be rational. If it is rational, it may or may not be true; and if it is irrational it must needs be untrue. All that is assumed is that the true is rational, and that all ideas are real. (The algebraic logical analysis is given in the following chapters.)

The true and the untrue pass imperceptibly one into the other, in a manner that the real and the unreal could not do except by a continuous creation as Bergson supposes. If this continuous creation presupposes a universal time which Bergson (erroneously) postulates, the theory of relativity raises many obstacles in his path, which though surmountable must be considered somewhat unbecoming, to his followers.

Both Hegel and Plato appear to have maintained that all ideas overlap, or interpenetrate. But if the *extension* of each exceeds that of its overlapping portion, the two ideas would accordingly not be regarded as identical. If A is B, and C is B, it does not follow that A is C, or C is A, yet A and C have just as much in common with each other as each has with B. This is a most important point, for it shows that in this respect, A, B, and C are identical, however much they might differ in all other respects.*

This identity in difference gives unity and plurality to all connexions between ideas.

§ 3

The modern conception of the inseparability of thought, feeling, and activity or will, as a fundamental principle in psychology; that is in cognition, affection and conation due to Herbart, forms the basis of the psychology of James Ward, who has developed the subject-object theory of Lotze with great logical precision. There is a certain aspect of Hegelianism with which it may be reconciled. This unity in the subject-object relationship of ideas as a metaphysical entity includes the psychological unity of thought, feeling, and activity. There is no thought without some vestige of feeling and activity; no feeling without some trace of activity and thought; no activity without some element of thought and feeling.

When we push this conception to its infinitesimal limits by the Hegelian methods, a procedure which subsequent considerations show to be legitimate, we arrive at the fundamental concept of mind-stuff in the elementary idea already emphasized.

The relationship that exists between *cognition* and *conation* will be shown to be a necessary result of the theory of relativity; but feeling or *affection* which is inseparable from both must be regarded as an idea connected with the sense of change in space and time. The metaphysical and physical diad involves a psycho-

* Thus if $A = qB$, and $C = q'B$, this means that $A = A_qB$, and $C = C_{q'}B$; so that A and C have B in common quite apart from the particular qualities q and q', since $AB = A_qB$, and $CB = C_{q'}B$. Hence A and C are identical so far as the quality B is concerned.

logical triad. Following Bradley the continuity and unity of thought or cognition is rendered possible only through feeling or affection.

Hence thought without emotion, though in accordance with the 'static thought' of the older logic of Aristotle, fails before the breaker of reality. 'Dynamic thought' is continuous, self-conscious, and alive. This 'mysticism' renders emotion an element of thought; and activity or will a function of all thought and action.

Psychical time or duration is not the same as physical time; since the latter is measurable only by reference to motion, and the unit of time in space: the former being the subjective rate of succession of ideas. The latter is, moreover, on the theory of relativity in a 'space-time continuum'. Two or more bodies may have the same physical time, but only under certain conditions, and then in general all bodies have different times according to their relative positions and velocities; further their times are, owing to the smallness of their velocity relatively to that of light, approximately though hardly ever actually the same.

If, however, the 'space-time continuum' of the monad is a purely idealistic conception, its space-time line would be subjective, such that if its time be altered its space relations would be altered too; for its physical time might in certain circumstances be identified with its psychical time or sense of duration; as in the case of a hypnotized subject; and even of a hypnotized crowd which, as is well known in India, can be made to think and feel in accordance with the desires and wishes or commands of a single individual. The photographic camera has failed to reveal any real physical effect in the so-called disappearing boy rope trick; and there is no doubt whatever that this is a delusion on the part of the crowd of witnesses around the performer, caused by some process of hypnotism that cannot deceive the photographic plate.

There are various methods by which this crowd hypnotism might be produced. The La-la-la, La-la-la, La-la-la . . . sort of song with which Indian conjurers try to entertain their audiences, has with a particular voice and intonation an effect not unlike that of M. Coué's formula 'Ça passe; ça passe, ça passe, ça passe . . .' repeated some twenty or thirty or forty, or a hundred and many hundred times till the audience becomes somewhat sleepy and highly sensitive to suggestion, particularly when the optic nerve of the observers is relaxed by the eye being turned upwards for some time in the effort to look up to see the disappearing boy.

The monotonous tone and greatly increased repetition of La-la-la, La-la-la's intensifies the effect over the Coué formula. It is easy to make experiments relating to the relative effects of repeating the same sound again and again, and those of varying sounds as in a formula, such as that of M. Coué. Moreover, the use of a very long pole in the first instance, when the audience round about are told to look up is a powerful aid as a hypnotizing means, just as a hen can be hypnotized by being made to look along the line in a tennis-court. Nevertheless, although I am not quite satisfied that all the best conditions for the most successful performance of the trick have been ascertained, nor do I press their explanation in this particular case, there is little doubt that an audience can be hypnotized; and that is all that concerns us here.* In the hypnotized state the sense of physical time becomes altered. The following experiences recorded (7 May 1874) by Professor W. Macneile Dixon (*The Republic of Letters*) are of particular interest in this respect. Tennyson wrote:

'I have never had any revelations through anaesthetics, but a kind of waking trance (this for lack of a better name) I have frequently had quite up from my boyhood, when I have been alone. This has often come to me through repeating my own name to myself silently, till at once, as it were, out of the intensity of the consciousness of my individuality, the individuality itself seemed to resolve and fade away into boundless being, and this not a confused state, but the clearest of the clearest, the surest of the surest, utterly beyond words, when death was an almost laughable impossibility; the loss of personality (if so it were) seeming no extinction but the true life.'

The repetition of his name when he was alone, as a necessary preliminary to the experience, shows that the effect was some kind of self-hypnotized trance. Nevertheless, the clearness of the vision, and the self-satisfaction with the harmony of being, is a feeling akin perhaps to that of certain forms of ecstasy in religious experiences such as those of St. Theresa, St. Francis of Assisi, St. Catherine of Siena and so very many other mystics; and though last, not least, of St. Paul himself on the road to Damascus! Other types of mystics like Swedenborg, and still more perhaps the Esoteric Buddhists and Theosophists, whatever the true explanation of their experiences might be, must at least be regarded as abnormal in their psychological states.† Eloquent emotional preachers have the same effect.

Experiences of this sort tend to make men Pragmatists, and so

* I do not propose to enter into the question of the correct explanation of this phenomenon, which is a side issue.

† William James, *Varieties of Religious Experience.*

regard truth as merely an intellectual satisfaction, depending upon the disposition of the individual; like beauty as the result of aesthetic emotion. The reality is in the feeling, quite apart from the truth of the contents of the experiences. The point is that only through emotion are the contradictory elements of thought reconciled and perceived as a reality which satisfies the understanding. It cannot be admitted with Eucken that activity is the sole means of attaining this sense of reality; since inactivity and even passiveness are often its most ready antecedents. The intense sense of joy and self-satisfaction aroused by vigorous physical exercise, by prolonged effort in rowing, or in mountain climbing, is a realization of being as a thing of value in itself. The incoherency of pure thought, the purposelessness of our existence; the whence, whither and why of all being, find a ready satisfaction in the feeling that connects thought with action in Eucken's sense; or that bridges the gap between the irreconcilable elements of thought in pure reason. The practical reason supplies in its stead, as the continuity of ideas; yet 'I have no word for it', said Goethe, 'feeling is all'. If this is not the whole truth, it is at any rate the principal factor in the fact of pure consciousness itself, whether in activity or passivity.

Space and time then are merely *a priori* intuitions of the understanding, and contradictory because not noumena but phenomena; as Kant has so clearly shown. The reality is in the feeling, and the thing *per se* is not what is perceived; only through feeling is its reality realized in thought. There and there only is the reality indisputable; the cause may be physical, our theories concerning it various, but the reality of the feeling is what cannot be doubted. This is the unifying principle in thought; without it, it is a mere formula in Aristotelian logic. But thought, so long as it is not pushed beyond the limits of personal experience, is quite adequate with action or will, to satisfy the demands of our nature; provided that they combine so as to fulfil the conditions of thought in the immediate present and surroundings, with approximate accuracy, sufficient to satisfy the feeling of activity as realized by the understanding.

Not in the Aristotelian but in the Hegelian logic does the healthy mind find its solace and repose; its comfort and intense satisfaction. Static thought is sterile and contradicted by reality. Dynamic thought is active, prolific and alive. The mathematical logic is a fabric of ideas; fit to grapple with any problem whatsoever; and competent to realize the best results, with critical and unfailing comprehension.

Just as the Hegelian dialectic is often the very essence of wit

by the subtlety, the clearness, and the precision of the distinctions; so perhaps is the 'Irish bull' of common parlance.

Two of the most familiar expressions in the conversation of rural Irishmen, in most matters, are: 'Is *that* a faact?' To which the almost invariable reply is, 'Of coorse to be exact, it is, and it is *not*'.

But there is no discontinuity in knowledge; ideas are its real substance; and its beginnings are its ends. 'Mind-stuff', bioplasm, matter, is endowed with qualities none other than ideas, without which they mean nothing and would cease to be. The *thing-in-itself* or *Ding an sich*, though not affording *sense data*, must possess qualities as possible *sense data*.

§ 4

That Nature is the development of an idea may be intelligible, when analysed. But that things are ideas, and science only a short-hand expression of the relationships between those ideas, is a thesis which at first sight might not be so readily conceded. Yet the method and results of this investigation all confirm this view. For the 'invariants' of our equations are in fact but Nature's laws: a particular class of which form the basis of Einstein's theory of relativity.

By the use of tensors, or vectors of a particular type, forming invariant relations which remain unaltered in different systems of coordinates he has arrived at the most astounding results which have lent themselves to experimental verification. His transformations may be arrived at similarly by what is called the theory of groups. But other and more extensive methods, some of them still far from perfect, and in some instances only in the embryonic stage, yet afford aid to the elucidation of fundamental problems: i.e. physics, mathematics, logic, metaphysics, psychology, and biology, not long since regarded as distinct, are found to merge gradually one into the other, like the straight line that returns into itself.

This will appear in the mathematical treatment of the subject. The analysis and subsequent synthesis which necessarily follows from it, though merely the result of the elimination of contradictories, constitute a process of solution of equations through various operations, usually of a simple kind, though occasionally in complicated problems the operations are of a highly intricate nature; but it is then that the method is found to be a powerful aid to thought, as the reasoning tends to become too involved for the unaided understanding.

Whether the method is all that is claimed for it here, the reader with some patience will be best able to surmise. Suffice it to say,

for the moment, that it has proved fruitful and stimulating for inquiry in attaining relations which, if not in every case beyond criticism, yet I believe them to be such as to lend themselves to further rectification and elucidation by those very same methods that cannot fail in time to check the accuracy of the results.

The thing-in-itself or *Ding an sich*, though not affording *sense data*, must possess qualities as possible *sense data* of perception; and as such retain an independent existence, independent of you and me no doubt, but yet its qualities are ideas as such in their very nature, else they would cease to be properties that any intelligent and rational being could ascribe to them. If here we reach the threshold of human knowledge, or of human ignorance, we reach at any rate the bed-rock of human reason, the denial of which must of necessity (and I use the word *must* most emphatically and deliberately) deprive us of certainty in all other matters whatsoever with which our faculties may be employed. Agnosticism in such circumstances betrays only indolence of mind.

Lord Russell assumes very rightly in his *Problems of Philosophy* that if bodies have *sense data* they must exist; but that it does not follow that if they exist they must have *sense data*. Quite true indeed. But if they exist they must have some properties or qualities of some sort; and such qualities mean nothing except as predicates of a subject, that is they are ideas related to other ideas; and can be nothing more or less than ideas. Hence if they exist without *sense data* as perceived by you or me they at least exist as ideas in themselves Notions, perceived however dimly by themselves as monads or else by some other mind in which they exist as ideas. *The realist says*, No. *The qualities apply only to the idea of the thing*, and *not to the thing itself*. In other words the thing itself has no qualities if it is not an idea, since subject and predicate must both be ideas; but a thing without qualities (which are not being perceived) may nevertheless exist. Such is the realist's standpoint. The idealist maintains that anything that exists must have some quality, else the statement that it exists becomes meaningless, or in other words nonsenical. Qualities are ideas, and exist only when perceived. This is contrary to the realists who assert that a thing may exist without being perceived, in which case it could not be said to have any qualities. The possibility of a thing existing without any qualities whatever seems to be a denial of reason; for existence is itself an idea, and therefore if it can be predicated of something as a subject that something is itself an idea, unless the existence is predicated only of the idea of the

thing and not of the thing itself, which it obviously is not.* Hence there can be no doubt that if words have any meaning realism other than idealistic-realism can have none; and idealism emerges triumphantly from the issue. The thing-in-itself need not be denied but it must ultimately be knowable. Whether the thing-in-itself is an idea or not, it yet must ever be associated with an idea as subject of which its qualities are the attributes. In fact the thing-in-itself if not an idea becomes a gratuitous assumption based no doubt upon bad induction and analogy or synthesis; but if not an idea it is unknowable and as such meaningless. Kant maintained that the noumenon was for ever beyond the reach of human faculty; but he would have had to admit that it was intelligible to some higher mind, or else impossible. 'Existence' has various meanings, see later, Chapter IV.†

> As we say, even if a thing could exist without any qualities whatever other than the fact that it exists, yet existence being essentially an idea, the predicate existence would apply only to a subject that is itself an idea, else the proposition could have no meaning.

Hence anything that exists must essentially be an idea whether it is thought of, or not. Though the ultimate subject A in an infinite series of terms A is B; B is C; C is D; D is E; . . . M is N; O is P should remain for ever beyond the reach of any intelligence, yet must it be an idea in the limit. A thing that is not being perceived may no more exist than the future can be said to exist. As we shall see, the future may exist in an idea. 'The permanent possibility of sensation' is all that can be affirmed. A thing may be brought into existence when it is perceived, and repeat itself in being recreated, as Bergson holds. It is upon this point that so many systems of metaphysics have been wrecked. Only the idealistic-realism of to-day has been able to keep afloat in the ocean of this great enigma. The confidence in human reason and the laws of Nature that we trust from the inferences by analogy, induction and scientific experiments, however, enable us to steer our way to the shores of exact knowledge and verifiable ground of so-called material phenomena which in the limit resolve themselves into 'invariants' of thought, otherwise ideas related in a particular manner.

In all the efforts to solve the problem of the relation of mind to

* It is held by many logicians that existence is not a predicate, but I do not agree, since logically existing things form a class distinct from non-existing ones.

† The proposition p is q may or may not be true. If it is true, it is true for at least one p and one q, otherwise it is meaningless. (See p. 176.)

matter there appears to be an absence of a clear perception that matter is not only mind, and therefore an aggregate of ideas; but that it partakes throughout in an infinitesimally small extent or degree of the elements of mind, as 'mind-stuff'; each unit possessing its subject and object, infinitesimally united by continuity in its own scale of being; bridging within itself that apparent gulf that an unintelligible dualism otherwise entails. Although the idea of a thing and the thing itself may be quite different they are none the less both ideas.

The background of Nature is an idea, i.e. the Logos, in the sense in which the word is here used. Its properties are the properties of matter, the subject of scientific investigation. If regarded from the old-fashioned realist point of view the physical phenomena may be interpreted on the assumption of its independent reality. But from the point of view of biology and psychology there is a wide difference. The properties of the substance we call matter when regarded as ideas can be investigated from the qualities these reveal in their ultimate empirical aspect. Just as inertia and force present themselves to the physicist as the basis of his equations, so do certain other qualities of matter appear to be empirically ultimate *concepts* to the biologist and the psychologist. And the properties of these *concepts* constitute the ground of scientific investigation on the basis of experience.

These ultimate concepts follow certain laws which are in fact the laws of association of thought or the elements of mind-substance or mind-stuff.

A general outline of this theory was given in my book *The Origin of Life* (1906) and developed in various articles from time to time; but never systematically worked out until presented in this volume. It is of interest to note, however, that many biologists maintain—for instance J. S. Haldane in his Presidential Address to Sec. K, British Association (1908); and in his *Essay on Mechanism, Life, and Personality* (1913)—the importance of the biological *concept*, returning to the idealism of Bishop Berkeley; to which Huxley, FitzGerald and others at the close of the nineteenth century were inclined to revert. Strictly speaking, however, it is to the idealism of Leibnitz in particular that we should turn.* And the powerful school of Lotze, Driesch, William Stern, and Becher in Germany in recent times follows this logical development of thought. The antithesis, however, is inevitable.

Notwithstanding James Ward's aversion to Hegel's logic, which

* I am indebted to the late Dr. Mahaffy for directing my attention to the study of Leibnitz in the original.

he described as 'an entirely fallacious system of logic',* the tendency has been in England on the whole towards Hegelianism. For leaving aside the great works of T. H. Green, Edward and John Caird, William Wallace and others at the end of the last century, we have as I say had in very recent years such influential thinkers as F. H. Bradley, J. M. E. McTaggart, and Bernard Bosanquet; Hegelians of different types, but none the less Hegelians. The severe onslaughts on Bradley from Thomas Case in the *Encyclopædia Britannica* represented the spirit not so much of the times as of the bygones of English philosophic thought.

Notwithstanding his great learning and profound insight into reality, it does not appear that Ward really appreciated Hegel at his proper value; a circumstance that might be accounted for by the spell of Lotze's personality, by whose influence he appears to have been carried away. Yet Lotze does not seem to have had that great fascination for all his students; since Eucken, another of his pupils, though not a disciple, does not mention him in his *History of Philosophy* until the sixth edition; a much severer stricture than Ward's judgement of Hegel. All three have secured a position of importance in the history of European thought. Viewing them now from a greater distance than contemporary thought permitted, our endeavour is to see what is good in each without disparaging the other. One thing seems certain that when critics differ so profoundly, it is time to approach the subject from some other standpoint, and view it afresh with newer methods; to attack the situation with more modern weapons; for the failures of the past do not necessarily, as popularly supposed, mean the complete absence of success in the future.

* Presidential Address to the Aristotelian Society, 1920.

CHAPTER II

ALGEBRAIC ANALYSIS

§ 1

AS has been stated, the use of algebraic analyses in logic was proposed by Leibnitz, but the idea was not taken up seriously until George Boole tackled the question with singular ability, and obviously without any knowledge of Leibnitz's notions upon the subject. The latter indeed, though original and ingenious, never reached a stage of sufficient maturity to entitle them to serious consideration on the part of the mere critic who could not see beyond his nose, and was blind to the possibilities of the conceptions thus initiated by the great philosopher and author of the 'Monadology' from which Kant in his earlier days as the disciple of Wolff sought his inspiration. It is unnecessary to enter here into Leibnitz's methods, which were left by him but crudely undeveloped. Only his 'Universal Characteristic' needs to be considered.

His object was to reduce or resolve all notions into their primitive factors; just as in arithmetic we should determine the least common multiple and greatest common measure; that is the least number that is divisible by, and the greatest number that will divide into a given set of numbers without a remainder; so in the case of ideas, or classes, in general the problem in the first instance is to determine the factors which are common to them, in their most primitive and irreducible form. And from the combinations of these factors to construct the fabric of thought that constituted the subject or universe of discourse.

Needless to say his critics viewed such a project with dismay and the most discouraging pessimistic apprehension.

In Boole's hands, however, they assumed a more consistent and comprehensive form. He invented a new calculus which lays the foundation of a science of logic, and constitutes the basis of a general method for the application of the general theory of probabilities. The chief application of this method, with which we propose to deal, is to the laws of thought; and these in a more or less different aspect from the recognized Canons of the Schools as worked out by Aristotle so elaborately in the *Organon*. Further, the logic of Boole itself, as he left it, and as his followers have expounded it, though not Aristotelian in its procedure, so far as being syllogistic in its methods, is nevertheless confined to what I have described as 'static' thought. This assumes that we are

dealing with absolute truth and absolute reality in which there is no room for contradiction; whereas in actual life and in the phenomena of Nature we are continually met with contradictions, as we have to deal with appearances and not with realities: with approximate truths rather than with absolute ones. Thus the antinomies of Kant, and the dialectic of Hegel lead us to a higher synthesis of the Boolian logic. The Hegelian logic admits of a symbolic treatment, as I shall endeavour to show. This logic is in fact the logic of 'dynamic' thought. When we have to treat with 'forms of intuition' such as space and time, rather than with realities of things in themselves, the law of non-contradiction does not actually apply. Thus the equations though somewhat more elaborate, nevertheless admit of manipulation as though we had to deal with ordinary propositions. But the system of ratiocination is none the less different. Only by a mathematical treatment could the intricacy of the problems dealt with be clearly and lucidly brought to light. The principles of contradiction and of excluded middle—namely, that 'Nothing can both be and not be'; and 'Everything must either be or not be'—combined with the principle of Identity, i.e. 'Everything is what it is' implying the identity amidst diversity in any quality common to any or all classes or individuals; these in all cases hold only for the same place and time. They are true however for absolute realities or things-in-themselves; and paradoxical as it may seem, we have seldom or ever actually to deal with such in real life.

Modern logicians like Peano and Burali-Forti in Italy, and Whitehead and Russell in England, lay great stress upon the difference between two such propositions as '*a* is *b*' and 'all *a*'s are *b*'s'; both of which according to the older logic would be regarded as of the same type. In one case, however, the subject is an individual, but in the other a class consisting of a group of individuals. Classes can be supposed to overlap, but individuals cannot interpenetrate. But the individual may be regarded as a group of members of different classes unified. Such a unity is of course a contradiction. And Russell thinks that much confusion has arisen, as no doubt it has, from the traditional logic between classes and individuals, and the inference that individuals can interpenetrate because classes can overlap. Russell says in *The Analysis of Matter*, p. 387:

'The only author I know of who has dealt seriously with its difficulties is Bradley, in whom, quite consistently, it led to a thoroughgoing monism, combined with the avowal that, in the end, all truth is self-contradictory. I should myself regard this latter result as a refutation of the logic from

which it follows. Therefore, while I respect Bradley more than any other advocate of interpenetration, he seems to me, in virtue of his ability, to have done more than any other philosopher to disprove the kind of system which he advocated.'

He did not of course overcome the difficulty of unity in plurality. The problem of the One and the Many might be said to have remained almost where he found it, notwithstanding the remarkable theory of inference he put forth. The same might be said of Bosanquet. But each contributed in a marked degree towards the elucidation of the subject in its more modern aspect. That x can be $xyzw$ involves no contradiction, any more than that a table may have four legs, and be of oak, stained a dark brown, &c. For when one sees an object and perceives its various qualities all at once, no contradiction may be implied. The individual object unites all the qualities in itself, but the various classes of qualities overlap: the object is common to all such classes. This is assuming of course that the object and qualities are one; wherein the apparent contradiction is supposed to lie.

Now the object is the common ground of all these various classes. And so long as it is regarded—as the idealists regard it—merely as the intersection of the classes, but not as something distinct from these qualities when combined, there is no contradiction; since it is merely the unifying principle between them that, as in Smut's 'holism', constitutes the one in the many. The unity, however, may be a class in itself and may be regarded as existing as an idea in which some only of the qualities are essential. Since, in the case of the Ego, the overlapping of classes of things that exist with that of things that are self-conscious, from which existence is inseparably connected, constitutes a class of necessarily existing things.

It is not proposed to enter in this work upon the purely formal aspect of logic as given by Whitehead and Russell in *Principia Mathematica*, and by the former alone in his *Universal Algebra*; or indeed to follow the exhaustive nomenclature of Burali-Forti in *Logica Matematica*; or to make use of Peano's *Formulario*. These have their utility and instructiveness in their own place. Schroeder's *Vorlesungen über die Algebra der Logik*, and Coutarat's *Algèbre de Logique* likewise have their merits and advantages. The treatment here is quite different and independent; directed expressly towards the particular physical and philosophical problems under consideration. We are dealing not so much with formal logic, as with logic as the foundation of all science and philosophy, which was Hegel's standpoint.

The simplification made in ordinary algebraic equations, by omitting powers higher than the first, renders the logical problems from an algebraic point of view much simpler. This simplification may be compared to that of descriptive geometry, which is concerned with the methods of representing solids and other figures in three dimensions by drawings in one plane. Descriptive geometry has also been defined by Whitehead as 'the science of a class of relations, each relation being a two-termed serial relation, as considered in the logic of relations, ranging the points between which it holds into a linear open order. Thus the relations are the straight lines, and the terms between which they hold are the points. The 'point', however, is that there is continuity between the points.

In the logic I shall for the most part consider, order does not at first enter into the calculations except when expressly stated. xyz is the same class as xzy, or yxz, yzx, zxy, and zyx. They stand for one and the same class. In geometry as in the phenomenon of time, order plays an essential part. When order makes a difference the classes so differing must be regarded as distinct. The same and yet not the same as the order makes a difference. It will be readily seen, however, that this complicates matters but does not confuse them. In the endeavour, however, to regard them geometrically much simplification may be attained. Groups and members of those groups may have different relations, and the transformations that convert one type into the other are somewhat of the nature of projection in geometry.

In some cases the commutative law does not hold, even if the associative law does apply. Thus $(xy)z = x(yz)$; but xyz may not be the same as xzy.

A system of algebra applicable to associative but non-commutative classes has been worked out by Mr. Dirac. He has extended it to Heisenberg's theory of matrices. This important work has attracted much attention of late; and undoubtedly marks one of the most important steps in physical science since Einstein's generalized theory of relativity. We must await further development of this theory which threatens to displace all others as a new 'quantum-mechanics'.

When both associative and commutative laws hold, the operations as regards the relationships of classes do so too, whether or not the principles of contradiction, excluded middle, and identity apply.

In Boole's system the law of identity is expressed by the equation

$$xx = x; \text{ or, } x^2 = x; \qquad (1.10)$$

which may be written,

$$x(1-x) = 0; \qquad (1.12)$$

so that the values $x = 0$, and $x = 1$, satisfy this equation. $x = 1$, then, is taken to mean the whole class of things or propositions represented by the symbol x, as the whole 'universe of discourse'; and $x = 0$ means that the class x is non-existent, x exists when x is 1.

Hence by $(1-x)$ is meant the class that is not-x; and then that which is both x, and not x, does not exist.

Further, since $x+(1-x) = 1$, 'Everything is either x or not x'. The plus sign indicates the disjunctive *or*; *also* the conjunctive plus as the case might be. It is usual to designate $(1-x)$ by the single symbol $\bar{x}$ so that

$$x+\bar{x} = 1; \text{ and } x\bar{x} = 0. \tag{1.13}$$

Again if y be any other class, we have likewise $y+\bar{y} = 1$.

Hence we obtain that $(x+\bar{x}).(y+\bar{y}) = 1$; (1.14)

therefore $x.y+x.\bar{y}+\bar{x}.y+\bar{x}.\bar{y} = 1.$ (1.15)

That is, the universe of discourse is composed of (1) x's that are y's; (2) x's that are not y's; (3) y's that are not x's; and (4) those that are not x's and not y's; i.e. things that are neither x's nor y's.

The equation (1) $x.y = 0$ means that 'No x is y'. (1.16)

(2) $x.\bar{y} = 0$ means that 'There is no x that is not y'; that is 'Every x is y'. (1.17)

(3) $\bar{y}.x = 0$, 'Every y is x'. (1.18)

It should be definitely stated whether x and y are taken collectively or distributively: classes or individuals may be distinguished by Latin or Greek letters, and relations or propositions by Capitals —this is clearly a matter of conventionality, that must, however, be made definite.

(4) $\bar{x}.\bar{y} = 0$, 'Everything is either x or y'; meaning by 'or' the Latin *vel* not *aut*; because the possibility of x being y is not excluded, i.e. that which is neither x nor y does not exist. (1.19)

If *aut* is meant then $x.\bar{y}+\bar{x}.y = 1.$ (1.20)

Here both $x.y = 0$, and $\bar{x}.\bar{y} = 0$; but if only $x.y = 0$, then

$$x.\bar{y}+\bar{x}.y+\bar{x}.\bar{y} = 1; \tag{1.21}$$

so that then that which is neither x nor y is possible.
In general we have

$$(x_1+\bar{x}_1).(x_2+\bar{x}_2).(x_3+\bar{x}_3)\ldots(x_n+\bar{x}_n) = 1. \tag{1.22}$$

Another method of expressing the proposition 'Every x is y' is as follows,

$$x = v.y; \tag{1.23}$$

where v is an indeterminate, representing 'some'. Therefore

$$x.\bar{y} = v.y.\bar{y} = 0, \text{ since } y.\bar{y} = 0. \tag{1.24}$$

Further $x.\bar{y} = 0$, or $x.(1-y) = 0$, may be written $x = x.y$. (1.25)

In order to obtain a familiar acquaintance with Boole's method, his original writings should be consulted, notably his *Laws of Thought*, already referred to. Many of his methods, however, admit of very great simplification; and this we propose to make in employing the system in the course of this work.

In a book such as the present, some knowledge of this calculus must be postulated in dealing with several of the questions to be discussed; although it is to be hoped that, except where otherwise stated, the explanation of the methods used will be found sufficiently clear to the ordinary reader equipped with an elementary knowledge of the subject. [N.B. The general reader may on the first reading omit the rest of this section unless he is already acquainted with the notation and fundamental principle.]

The proposition $p \supset q$: p implies q, or all p is q, is written in the Booleian logic $p\bar{q} = 0$, i.e. there is no p that is not q, or p is always q. Whitehead and Russell in *Principia Mathematica* would express this $p \supset q : \supset (\bar{p}+q)$. Now this is quite true; but it also is true that

$$p \supset q : \supset (q+\bar{p}\bar{q}) : \supset (pq+\bar{p}). \tag{1.26}$$

For $p\bar{q} = 0$ means $pq+\bar{p}q+\bar{p}\bar{q} = 1$, (1.27)

that is of course

$$q(p+\bar{p})+\bar{p}\bar{q} = 1 \text{ or } pq+\bar{p}(q+\bar{q}) = 1, \tag{1.28}$$

and since $p+\bar{p} = 1 : q+\bar{p}\bar{q} = 1$ and $pq+\bar{p} = 1$. (1.29)

In other words if p is q then either q or *not-p*: but it may be q, whether p or *not-p*; or else *not-p* and *not-q*, which is equivalent to either both p and q or else *not-p* which may or may not be q. I prefer to give the three alternatives, namely, either p and q or q and not p, or else neither p nor q.

The Propositional function x is y or $x\bar{y} = 0$ may not be true unless y is a, i.e. $y\bar{a} = 0$ when a is some particular value of y. But it is expressed by the single product $x=xya$, i.e. $x(1-ya) = 0$ or $x(y\bar{a}+\bar{y}a+\bar{y}\bar{a}) = 0$ which consists of three separate propositions

$$\begin{cases} xy\bar{a} = 0 \\ x\bar{y}a = 0 \\ x\bar{y}\bar{a} = 0. \end{cases} \tag{1.30}$$

The full solution of the equation is

$$xya+\bar{x}(y+\bar{y})(a+\bar{a}) = 1. \tag{1.31}$$

$p \supset q$ $p \in q$ $p\bar{q} = 0$ $p = 1 \supset q = 1$ $q = 0 \supset p = 0$ are due to Peano.

$\supset$ = universal class
$\in$ = individual member of a class
p is a sufficient condition of q
q is a necessary condition of p.

$\cap$ = logical $\times$ $\cup$ = logical $+$

$p = v.q$, $p = pq$. $\therefore pq = vq$, and $p\bar{q} = 0$.

It is easy to see that
$$p+q = 1-\bar{p}\bar{q}, \tag{1.32}$$
or in Peano's notation
$$p \cup q = -(-p \cap -q); \tag{1.33}$$
also
$$pq = 1-(\bar{p}+\bar{q}), \tag{1.34}$$
or
$$p \cap q = -[-p \cup -q]. \tag{1.35}$$
In such cases the superiority of the Booleian or ordinary algebraic notation is obvious.

For matters of definition the Peano notation is more suitable. A few remarks on Peano's notation may be useful.

$\in$ is used for *is* in individual propositions.

$x \in y$ as in 'Socrates is a man'. $1 \in N$, one is a number.

But $\supset$ for universal propositions.

$x \supset y$. 'Englishmen are Europeans'.

$\ni$ is used for 'that which' or 'such that'.

$x \ni .\ x \in Np$. x is such that x is a prime number.

ι stands for 'equal to' and $\jmath$ for 'the' or the only member or a class with one member.

Thus $\iota\, 2 = Np \cap 2N$ and $2 = \jmath.Np \cap .2N$.

2 is the first prime number multiplied by 2.

Then $\cap$ represents logical multiplication.

Then $\cup$ represents logical additions in Boole's system $+$.

$\supset abc$ means it follows that whatever a, b, and c may be.

The fundamental logical inferences are

$$\left.\begin{array}{l} (1)\ a \supset b.b \supset c \supset_{a,b,c} : a \supset c. \\ (2)\ x \in a.a \supset b \supset_{a,b,c} : x \in b. \\ (3)\ a \in b.b \in c \supset_{a,b,c} : a \in \cup c. \end{array}\right\} \tag{1.36}$$

In Boole's system

$$\left.\begin{array}{lll} (1)\ a = v.b & b = vc & \therefore a = vc. \\ (2)\ x = xa & a = v.b & \therefore x = xvb. \\ (3)\ a = ab & b = bc & \therefore a = abc. \end{array}\right\} \tag{1.37}$$

There is thus a marked difference between the symbols $\in$ and $\supset$.

The symbol $\supset$ is transitive. A relation is said to be 'transitive' when the object is in such a relation to a second, and this in a similar relation to a third, it can be concluded that the first is in such a relation to the third.

The relations
$$\left.\begin{aligned} x = y, y = z &\supset x = z \\ x > y, y > z &\supset x > z \end{aligned}\right\} \qquad (1.38)$$
are transitive.

Instead of two universal propositions, an inference may be drawn from one singular and one universal. But from two singular propositions no conclusion follows, and the symbol $\in$ does not stand for a transitive relation. For instance from $x \in y, y \in z$ it cannot be inferred that $x \in z$. Peano gives the example, Peter and Paul were Apostles. The Apostles were twelve, or, Peter and Paul $\in$ Apostles. The Apostles $\in$ (class of 12 men). It does not follow that 'Peter and Paul were twelve'. Russell observes similarly that 'All men are numerous. Socrates is a man .·. Socrates is numerous'. If we had substituted humorous the argument would have been valid, because one is taken collectively and the other distributively. The old logic does not make this distinction. But it is a matter of $x = xy$, $x = vy$.

Two classes are similar when there is a one-one relation between their members, as between the classes of husbands and wives in a monogamatic state. Russell then defines *the number of a class* as *the class of all those classes that are similar*. The number two is the class of all couples. Thus 'A number will be a set of classes such as that any two are similar to each other, and none outside the set are similar to any inside the set'. i.e. I prefer to put it, a number (in general) is any collection of which any two are similar to each other, &c.

As we have said, Peano postulates three primitive conceptions as the basis of Arithmetic: o (zero), N_0 (number), $+$ (plus or successor); and five primitive propositions:

(1) $o \in N_0$ (zero is a number and N_0 is a class of which zero is a number).

(2) $a \in N_0 \supset a+ \in N_0$ (if x is N_0 then every successor of a is a number).

(3) $s \in \text{Cls.}, o \in s : x \in N_0 \cap s \supset_x, x+ \in s \supset N_0 \supset s.$ (1.39)

[Principle of induction: If s is a class and o a member of s; and if x is a member also of the class s, then all members belong to s.] This is equivalent to saying that any property which belongs to

o and to the successor of every member possessing this property, is a property of all members. So that

$$x \in N_0 \supset_x . \qquad x = (x+1), \in N_0. \tag{1.40}$$

(4) $a.b, \in N_0, a+ = b+ \supset a = b.$

If the successors of two numbers are equal, the numbers are equal, or 'two numbers have the same successor' (Russell).

(5) $a \in N_0 \supset a+ - = 0$

where — stands for 'not'; i.e. the successor of a number is not zero.

The successor of a number is not zero. This enables us to define infinity.

$$n = n+1 \supset n \in -N_0 : n \supset \aleph_0. \tag{1.41}$$

If n is the same as $n+1$ then n is not a number but is infinite. Nevertheless we do talk of infinite numbers, but they are not natural or inductive numbers, and ought not to be regarded as numbers at all as the principle of induction (3) does not apply to them.

If x is y sometimes or some x is y: then $vx = v'y$ or $x = \frac{v'}{v} \cdot y$.

Put $z = \frac{v}{v'}$, then $x = zy$.

$$\therefore x\bar{y} = 0 \qquad x\bar{z} = 0 \qquad yz\bar{x} = 0$$

$$\therefore (xy + \bar{x}\bar{y})z + \bar{x}\bar{z}(y+\bar{y}) = 1$$

and* $z = vv' + \bar{v}\bar{v}'$ and $x = (vv' + \bar{v}\bar{v}')y + \bar{v}(v' + \bar{v}')\bar{y}$. (1.42)

According to Peano's notation $x \supset y$ means All x's are y's when x is a class; but $x \in y$ is x is a y, x being an element of the class y.

In either case $x\bar{y} = 0$ or $x = \frac{0}{\bar{y}} = \frac{0}{0} \cdot y + \frac{0}{1} \cdot \bar{y}$,* (1.43)

which is equivalent to $x = v.y$: v being either a class of elements or an element of the class y. Here Peano's nomenclature is of importance. *In Boole's method, however, as in the older logic, if one means an element one must say so, and use a special symbol for the purpose.*

In general $x\bar{y} = 0$ or $x(1-y) = 0$ means $x = xy$. This is not the same as $x = vy$. Because $v = \frac{x}{y} = \frac{1}{1} \cdot xy + \frac{0}{0} \cdot \bar{x}\bar{y}$, whereas

* See Chapter III.

$$x = \frac{x}{y} = \frac{1}{1} \cdot xy + \frac{0}{0} \cdot x\bar{y} = 1 . xy \text{ since } x\bar{x}\bar{y} = 0. \qquad (1.44)$$

It would appear therefore that $x \in y$ is equivalent to $x = xy$ and $x \supset y$ to $x = vy$. But either can be transformed to $x\bar{y} = 0$. By eliminating v from $x - vy = 0$ and putting $v = 1$, $v = 0$ we obtain

$$x(x - y) = 0 \quad \text{or } x - xy = 0, \quad \text{i.e. } x(1 - y) = 0. \qquad (1.45)$$

If every a is b, and every b is c, and every c is d, and so on, until we come after a series say to z; then every a is z. If now every z is a, then z and a become identical; and so do all the other terms of the series. So that $a = b = c = d = \ldots w = x = y = z = a$.

If, however, it is found that although as before a is b, &c., yet z is not an a; i.e. $z . a = 0$; then since from the series $a\bar{z} = 0$ as a is z, and yet if a is not z then a is not a. But we are given that a is b, &c., so that it cannot be true that z is not a, i.e. $za \neq 0$; therefore only *some* z is a; namely the a which is b, which is c, &c.

If we call this $v.z$. Then $vz . \bar{a} = 0$. (1.46)

Let us put vz = some z that is a; and $v'z$ = some z that is not a, i.e. $vz . \bar{a} = 0$; and $v'z . a = 0$. We may for simplification designate $x = vz$, and $y = v'z$. So that $x . \bar{a} = 0$, and $y . a = 0$. Moreover $x . \bar{z} = 0$, and $y . \bar{z} = 0$.

Consequently

$$a . \bar{y} . \{x . z + \bar{x}(z + \bar{z})\} + \bar{a} . \bar{x} . \{y . z + \bar{y} . (z + \bar{z})\} = 1. \qquad (1.47)$$

$$\text{If } z . \bar{a} = 0; \quad a . \bar{y} . \{x . z + \bar{x}(z + \bar{z})\} + \bar{a} . \bar{x} . \bar{y} . \bar{z} = 1. \qquad (1.48)$$

The identity then is in the particular a's which are x's and z's; and $x . \bar{z} = 0$, and $z . \bar{x} = 0$.

$$\text{If } z . a = 0; \quad \bar{x} . \{a . \bar{y}\bar{z} + \bar{a} . y . z + \bar{y} . (y + \bar{y})\} = 1. \quad \therefore x = 0 \text{ or } vz = 0.$$

Thus in the series or sorites beginning with a is b, &c., if we ultimately find that say z is not a, then some a is not a. Likewise since a is b, some b is a, and since b is c, some c is b, &c., so that finally some z is a, namely only the *some* a*'s that are the common factors throughout the whole series*.

It is there merely a continuous identity throughout; such in fact is the Bradleyian theory of inference, and the Hegelian notion of the idea and its otherness, which returns back into itself. It would be fallacious to assert that all z is a so as to make a, b, c, &c. ultimately identical, as some have tried to do. Such a procedure would undoubtedly be entirely illogical and unsound.

Some general Principles.

According to this method, there are two ways of proving that 'Every x is y' or that $x.\bar{y} = 0$. Consider two classes z and w. Let z be the alternative of y; so that if a thing is not y it is z. (This does not exclude the possibility of its being both.) Now if x is not z then it must be y. Hence if we could determine something which is not x, but which is an alternative to y, it would follow that x was necessarily as a consequence y. Of course there might not be such a thing as z possible, in which case x might not be y; but if there be such a z then x must be y. That is to say if $xz = 0$, and $\bar{y}\bar{z} = 0$; then $x\bar{z}y + x\bar{x} = 1$; $\therefore x\bar{y} = 0$. Similarly if w be a middle term between x and y; then if 'Every w is y; and every x is w; it must follow that every x is y'. That is if $w\bar{y} = 0$; and $x\bar{w} = 0$; then $xwy + \bar{x} = 1$.

In general $x\bar{y} = 0$ if $x\bar{y}(z+\bar{z})(w+\bar{w}) = 0$, so that each of the four terms vanishes it matters not how, i.e.

$$x\bar{y}zw = 0; \quad x\bar{y}z\bar{w} = 0; \quad x\bar{y}\bar{z}w = 0; \quad x\bar{y}\bar{z}\bar{w} = 0. \quad (1.49)$$

These four terms will vanish if either of the two factors we have been considering vanishes; or else if their combination does so. The latter case obviously involves three propositions, instead of two as in the preceding ones, to prove that $x\bar{y} = 0$.

Thus if $x\bar{z} = 0$; $z\bar{w} = 0$; $w\bar{y} = 0$; then $x\bar{y} = 0$. This is of course the well-known sorites.

There are other ways, however, in which we might prove that $x\bar{y} = 0$. Thus it might be that $xz = 0$; i.e. 'x is not z'. But $w\bar{z} = 0$; i.e. 'Every w is z'; and $\bar{w}\bar{y} = 0$, 'That which is not w is y'. Therefore $x\bar{y} = 0$; 'x is y'. In other words: x is not z. But w is z, therefore that which is not z, and therefore x is not w. But we are given that if a thing is not w it is y, $\bar{w}\bar{y} = 0$. Consequently x is y.*

§ 2

Before going further into this matter it is desirable to explain

* In the paradoxical quotation from Shakespeare given at the beginning of this book a symbolic analysis at once renders his meaning clear. He says that

> 'Nature is made better by no mean,
> But Nature makes that mean; so over that art,
> Which you say adds to nature, is an art
> That nature makes . . . This is an art
> Which does mend nature, change it rather;
> But the art itself is nature.'

If we put N = Nature, or nature, which are apparently used synonymously.
x = things made better by a mean. Then $Nx = 0$.
y = things that make a mean. i.e. $N\bar{y} = 0$.
A = Art.

a curious fallacy already alluded to which the intelligent reader will perhaps have anticipated in the simple case of the Booleian equation representing the law of contradiction in the very simplest case. It is in fact what is now known as the *Russell contradiction*.

We know that in general $x.y = 0$ means that 'No x is y', or in other words 'Every x that is y does not exist'.

Now if $x = 0$, or $x = 1$, the two solutions of the equation of contradiction $x(x-1) = 0$, it is easy to see that we may very readily find ourselves landed in a contradiction. For if (1) $x = 0$; then since $x(x-1) = 0$, $x = x^2$, and therefore $x.x = 0$; that is x which is x does not exist, which actually means that x is not x.

Similarly (2), if $x = 1$, we have $1-x = 0$; therefore

$$(1-x)(1-x) = 0;$$

consequently not-x is not not-x or not-x is x.

Now what is the meaning of this? It means that when $x = 0$ we cannot say that it is identical with itself. In fact $x = 0$; no more holds strictly for the laws of thought than the imaginary solutions we have to consider, of the identity $x^n = x$. (1.50)

Russell's contradiction consists in considering 'the class possessing as members all classes which are not members of themselves'. If w is this class, and x is one of its members, then this is equivalent

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z = things that add to Nature. $A\bar{z} = 0$.
w = things that Nature makes. $q\bar{w} = 0$.
p = things that mend or change Nature. $q\bar{p} = 0$.
q = things over or beyond that art, which adds to Nature. $q\bar{N} = 0$, and $N\bar{q} = 0$. $qA = 0$.

$\therefore\ N\bar{x}y+\bar{N} = 1;\ Az+\bar{A} = 1;\ qwpNA+\bar{q} = 1.$

$\therefore\ qwpNAzy\bar{x}+\bar{q}\bar{N}(Az+\bar{A}) = 1.$

Hence,—over or beyond that art which adds to Nature is that art which Nature makes; this mends or changes Nature; but it is Nature no less than art itself; though it adds to Nature; and whilst Nature makes a mean, yet Nature is made no better by that mean. There is, however, or there may be, an art which is not over or beyond that art which adds to Nature; and it is not Nature, but it adds to Nature. That which is not art is not Nature; nor is it over or beyond that art which adds to Nature. In a word there are two kinds of art, that which is over or beyond that art which adds to Nature, and is Nature itself; and that art which is not Nature, but none the less adds to it. Quite a number of other inferences can be drawn from the equation, but the reader can make them for himself.

If $A\bar{N} = 0$ as is perhaps implied

$$qwpNAzy\bar{x}+\bar{q}\bar{N}\bar{A} = 1.$$

And all Art is Nature and Nature Art. All Arts overlap or become identical. There is a difficulty in making sense of the whole statement unless something transcendental is implied to the effect that Nature and Art are one, and are overlapping of different types, which is a Russell contradiction.

to saying that $x\bar{w} = 0$; and at the same time $x.w = 0$. But since in general we have $(x+\bar{x}).(w+\bar{w}) = 1$; or $(xw+\bar{x}w+x\bar{w}+\bar{x}\bar{w}) = 1$; $\therefore \bar{x}(w+\bar{w}) = 1$; i.e. $x.\bar{x}.(w+\bar{w}) = x$; or since $w+\bar{w} = 1$, then $x\bar{x}=x$; i.e. $x = v.(1-x)$ or x is not-x; in other words $x = 0$.

Likewise if $w\bar{x} = 0$, and $wx = 0$, then $\bar{w}(x+\bar{x}) = 1$; or, $\bar{w} = 1$, and $w = w\bar{w}$, or $w = v.(1-w)$, that is w is not-w or $w = 0$. A class and its members are different types.

This is equivalent to the problem of 'the one and the many', for in what sense can it be said that the members that compose a class are to be identified with the class itself; since this is what it comes to. The explanation is that whether we take x and w to be classes or relations, i.e. propositions, they are absolutely meaningless if supposed to be absolutely true; and that only if they are not absolutely true can they have any meaning whatsoever as an approximation to the truth.

Such contradictions are worth noting; but it is obvious that they arise only because the law of identity cannot be applied to the non-existent; or to that which is not absolutely true.

The 'Russell contradiction' was pointed out, apparently for the first time as a mathematical paradox, only about twenty years ago, although it formed the fundamental problem of 'the one and the many' in philosophy throughout the history of philosophic thought. Many scientific papers have been written on it during the past two decades, from a mathematical point of view, and much difference of opinion still exists as to the correct solution of the problem.

A group that contains all possible groups as elements must itself be an element of the group, since it is a group containing all other elements.

Let $x_1, x_2, x_3, \ldots x_n$, be any number of elements; and $y_1, y_2, y_3, \cdots y_n$ all groups containing x_1, x_2, x_3 ... x_n but not themselves. And let $z = \phi(y_1, y_2, y_3, \ldots y_n)$. Then z either contains itself or it does not. If it does contain itself, it clearly does not contain itself, for it is a group of groups that do not contain their elements. But if it does not contain itself it must contain itself since it is a group of such groups. Hence either supposition leads to a contradiction as z either contains itself or it does not do so.

Russell's solution appears to be that a group is not an object in the sense that the elements are said to be. The genus is not a species of itself. But the identity of subject and object, universal and particular, genus and species in the notion or concept forms the basis of the whole Hegelian philosophy.

To avoid the contradiction it would appear to be necessary to

assume that in every case the group of elements, though an element, must be regarded as different from the element; in other words that a group of self-conscious selves may be a self-conscious self, but different from the individual selves. The universal is a particular. The universal must be a particular member of the class it represents, yet different from each separately. A class would thus appear to be identified with at least one of its members or elements.

There are many interesting applications of the principle underlying Russell's contradiction. The apparent inconsistency between unity and plurality is shown by Grelling's paradox. The word 'short' for instance is what it signifies, short; whilst 'long' is not what its meaning indicates. 'English' is English, and 'Deutsch' is German. 'Dreisilbig' is trisyllabic, and means what it is. Such words may be called 'autological'; whilst those like 'long' which are not what they mean, 'heterological'. The question is whether the word 'heterological' belongs to one class or the other. If it is autological, then it is what it means, i.e. heterological. But if it is heterological then it is precisely what it means and therefore autological.

The solution of the difficulty is that words do not consist of one or other of the two classes; but may belong to a third class that apart has two meanings, as both autological and heterological at once.

If w = words generally; w_1 = the word 'heterological'; a = autological; h = heterological.

Then since $$(w+\bar{w})(w_1+\bar{w}_1)(a+\bar{a})(h+\bar{h}) = 1,$$

and we have $w_1\bar{w} = 0$, and $w_1h\bar{a} = 0$; $w_1a\bar{h} = 0$. (1.51)

Therefore $$w_1w(ah+\bar{a}\bar{h})+\bar{w}_1 = 1.$$

The word 'heterological' therefore is either both heterological and autological at once, or it is neither one nor the other in the strict sense. It must form a class apart. Yet this does not get over the difficulty that there are contradictory elements in the third class ah. The word may be heterological in name, but autological in fact. But the solution is of the nature of a Hegelian synthesis.*

This is dealt with by Whitehead and Russell in their theory of 'Types' (see *Principia Mathematica*, Chap. II). The line I take up

* The words 'not-predicable' would afford a similar paradox, since to be not-predicable is itself to be predicable. Another familiar illustration is the statement 'I am lying', which if true means that I am not lying, and therefore that I am lying, and so on *ad infinitum*.

The village barber who is the only person in the village who shaves everybody that does not shave himself and nobody else, must, if he shaves, shave himself too; and is therefore a self-contradiction unless he is a non-entity, which his customers would hardly admit.

is in accordance with Meinong's. It is more general. A 'golden mountain', 'a unicorn', or 'a round square' are nominally possible, but not logical possibilities. The theory of 'types' involves categories or primitive propositions. No square is round. That a Perfect Being cannot be non-existent is self-evident. That there may be no such Being except in imagination is contradicted by the fact that if Perfect he must exist and cannot be merely imaginary. (See Chapter IX.)

The identity of subject and predicate in a judgement and the difference between the member of a class or group and the class or group itself renders a contradiction inevitable, if the subject is a member of a class, and the predicate a class or group. This of course is obvious. For if the subject S is a member of a class P, and it could be said of it that S is P, then since the predicate is a class distinct from any one of its members, and the subject is a member, it follows that the subject is an impossible thing. In other words the subject is not the predicate, or S is not P. That is to say that S is, and is not P.

This is assuming that a class cannot be identified with any of its members. But it may not be true that no member of a class is itself a class, or conversely that no class is a member of another class. Hence, provided a member of a class can be identified with the class no contradiction is entailed. The judgement then, however, does not become an identity.

Thus if x = subject; y = predicate; m = member of a class; c = class.

Then (1) $x\bar{m} = 0$; (2) $y\bar{c} = 0$. (1.52)

In general the universe of discourse is

$$(x+\bar{x}).(y+\bar{y}).(m+\bar{m}).(c+\bar{c}) = 1, \tag{1.53}$$

and therefore $\{xm+\bar{x}(m+\bar{m})\}.\{yc+\bar{y}(c+\bar{c})\} = 1.$

Now if $m.c = 0$, and $x.\bar{y} = 0$, we have

$$\bar{x}.[\bar{y}m\bar{c}+\bar{m}\{yc+\bar{y}(c+\bar{c})\}] = 1, \tag{1.54}$$

so that $x = 0$; or S is impossible.

But if $m.c$ is not $= 0$, then if $x.\bar{y} = 0$, we get

$$xmyc+\bar{x}(m+\bar{m}).\{yc+\bar{y}(c+\bar{c})\} = 1. \tag{1.55}$$

Hence x can be y so long as m is not incompatible with c. It is not necessary, as many imagine, that every member should itself be a class, or that every class should be a member of another class,

or identical with themselves. All that is essential is that the members that are x's and certain classes which are y's should be identical.

The synthetic judgement x is y is not possible unless m is not incompatible with c. That is *there must be an* a priori *intuition of the understanding which renders m and c simultaneously possible.*

§ 3. *The Simplification of Boole's Method*

So far we have had to deal with comparatively easy logical arguments that involved quite simple eliminations as we went along. These conclusions could in fact have been arrived at by the familiar methods of Stanley Jevons and his followers; though based upon the general principles laid down by Boole. These algebraic methods of reduction and elimination are frequently extremely laborious, as well as often unnecessarily complicated. Jevons's method is extremely mechanical. But the system of expression I have demonstrated, though based upon the same principle of elimination, has the advantage of giving a clear view of the situation at each step of the argument; while it also admits of actual reasoning in the process of elimination, as in the more refined algebraic processes. The extension to the Hegelian dialectic is, however, new; and the possibility of carrying the Booleian system beyond the ordinary static or Aristotelian laws of thought has, so far as I am aware, never been previously shown to be possible by algebraic means. Nor has the significance of the more general algebraic principles in their relation to logical procedure ever been perceived or recognized.

It has not been generally understood, however, by Boole's critics, notably by Jevons himself, that there are problems that admit of simpler solutions by Boole's method than by any others hitherto proposed.

By the exhaustive method of classification, i.e. if there are two classes x and y, then

$$(x+\bar{x})(y+\bar{y}) = 1, \tag{1.56}$$

in other words

$$xy+x\bar{y}+\bar{x}y+\bar{x}\bar{y} = 1. \tag{1.57}$$

If none of these classes vanishes, that is, if there is no general proposition true about any one of the four, then the Hegelian dialectic becomes operative, whatever x and y might be.

From the second and third terms a synthesis is formed in the union of the two mutually excluded terms in the first, which again in the fourth is excluded in the negation of both.

With a third class z, we have

$$(xy+x\bar{y}+\bar{x}y+\bar{x}\bar{y})(z+\bar{z}) = 1. \tag{1.58}$$

Students of Hegel find it extremely difficult at first to grasp the fundamental principle in the dialectic, that every proposition admits of being contradicted. The contradiction is in fact possible and necessary only because either no proposition fully expresses the whole truth, or else because reason is itself infinite. If reason be infinite, then the truth could not be expressed in a single proposition, but would be contained also partly in its contradictory, from the negation of which it would once more revert into itself, as a notion or concept; and from the union or identity of the process a higher synthesis is formed, which may again be contradicted and so on. . . . It will be seen then that, however much Boole may have disliked Hegelian logic, his own was in truth the real passport to it. The unity is in the whole universe of discourse.

A definite proposition such as $x.\bar{y} = 0$ could not be reconciled with its contradictory $x.y = 0$, unless $x = 0$, for then $\bar{x}.(y+\bar{y}) = 1$. But as being and not-being may be regarded as the same when merely viewed in the abstract, and not in the concrete, then if $x =$ being; and $(1-x) =$ non-being; if being is all that *is*, $x = 1$; and $(1-x) = 0$. But if $0 \times 0 = 0$, which it is not, then

$$(1-x).(1-x) = 0.$$

Therefore nothing is not nothing; i.e. nothing is something. Likewise non-being that is non-being cannot be; i.e. non-being is not non-being; i.e. non-being is being; or being and non-being are the same. (See p. 70.) But $x.(1-x) = 0$; i.e. being is not non-being: a distinct contradiction. As has already been pointed out, we have no right to ascribe identity to a nonentity. It is true that $x.x = x$; but if x is 0, then x becomes not-x; since $x.x = 0$; i.e. x is not x. But this is true, approximately, of the infinitesimal, where x is not absolutely $= 0$.

The same result or contradiction ensues even if $x = 1$, if we apply the principle of identity to the negation of x. For the idea which is identical with itself admits of an identity in an infinite number of ways. $x^n = x$. And if n is infinite it admits of an infinite number of solutions. Thus the number of times that a negation is performed by the process of dialectic affects the nature of the ultimate identity, which contains within it the idea of change in identity, as the source of difference in identity or origin of variation, as well as the general principle from which the idea of a 'space-time continuum' of the Minkowski-Einstein type may be evolved as a particular case of the general laws of thought and consciousness.

If $y = x^n$, then y is a function of x; i.e. a logarithmic function. y then represents the real value of x as though it were recorded in a cinematograph film by reflection. Not only in space and time,

but in the actual process of thought, whether in the realization of the 'otherness' or 'identity' of the Idea, is variation an essential concomitant of thought. (*Vide* p. 195 *et seq.*)

As I have endeavoured to show (*vide* p. 144 *et seq.*), the power n may be regarded as representing the *intensity of the idea*, perception or sensation; and y, of which it is the logarithm to the base x, may be regarded as representing the *intensity of the stimulus*. In any case we see that the feeling accompanying an idea is represented by n.

Hence when feeling is associated with an idea, and only when it is so associated with it, the laws of contradiction and excluded middle terms do not apply; and only when feeling is absent do the laws of the Aristotelian logic strictly apply. The Hegelian logic is thus explicable on the assumption that thought is accompanied with an element of *intensity* which corresponds to feeling. Thus cognition, affection, and conation become one; for we have shown (pp. 48, 52, 55) that cognition and conation are one as the result of the relativity of time; so that affection and conation now become inseparably connected in a manner that could at first sight have been by no means obvious; for affection is, except in the case of abstract ideas, the invariable accompaniment of thought.

The process of differentiation is itself the outcome of the logical identity when thought ceases to be abstract and acquires the element of *intensity*, the equivalent of feeling or affection in the technical sense. Now this was precisely the very element that we had been in search of throughout our researches as the unifying principle in the phenomena of Nature and of mind. F. H. Bradley has shown that feeling, on other grounds, forms an essential part of the unity of thought in the grasp of reality. The discrete elements of thought cannot in fact be united into a continuous whole without the fusing influence of feeling. Goethe vaguely felt this, though he knew not clearly what he meant when he exclaimed, 'I have no word for it. Feeling is all!' And if that feeling that fuses and unites subject with object be harmonious, as in truth it should be and in fact it is in the full realization of life, what is it, in its cruder forms, but sympathy and fellow-feeling; and in its higher forms but that love, as McTaggart held, which in its full realization attains to the perfection of reality as the ever-permanent and ever-true, as the adequate expression of thought and activity?

Neither action nor thought ever acquires to the level of reality when feeling can claim no part.

On the other hand, James Ward, following the track laid down by Lotze, has shown that the psychological unit is a trinity of

thought, feeling, and action. How or why this should be he did not attempt to explain. But the unity of physics, psychology, and logic, using the latter in its most modern sense, as the mathematics of classes of ideas, gives the full conception of reality as one consistent and harmonious whole. Thus the continuity between the discrete elements of thesis and antithesis is rendered possible through the conciliating influence of feeling.*

Nothing can be clearer than that if x be a class of proposition, and $\bar{x}$ its contradictory or negation, then $x+\bar{x} = 1$. Hence if this is identical with itself, $(x+\bar{x})^n = 1$. The expansion of which by the binomial theorem yields a number of terms of combinations of x and $\bar{x}$ in varying degrees of relative intensity, assuming of course that $x.\bar{x} \neq 0$, so that continuity exists between x and its opposite.

The negation of dialectic is not as might be supposed a mere continuous passage from one idea to its opposite, but a series of all possible graduations of combinations of relative intensity of x and not-x.

This is the converse of Dedekind and Cantor's theory of the infinite as applied to unity and identity—rather than to infinity: orders of identity and orders of infinity are analogous, but we are more familiar with unity than with infinity. In a later chapter we shall discuss the modern idea of the infinite as propounded by Russell and his school. As great an advance has been made in this direction since the beginning of the century as in Physical Science itself.

* Here surely there is an overlapping of ideas of different types.

CHAPTER III

GENERAL PRINCIPLES

MACAULAY asked who ever heard of a reaction to Taylor's theorem? Russell says in *Mysticism and Logic* that it has been overthrown. Yet I venture to reconsider its application to logic and particularly Hegel's logic. If I am wrong I can be put right. We assume provisionally that types do overlap.

In general any function of x may be expanded in a series of ascending powers of x, as follows:

$$f(x) = a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots,$$

and similarly any function of $(x+y)$:

$$f(x+y) = a_0(x+y)^n+a_1(x+y)^{n-1}+a_2.(x+y)^{n-2}+\dots.$$

Hence expanding by the binomial theorem, and arranging in powers of y, we get

$$\begin{aligned} f(x+y) = {} & (a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots)+ \\ & +(na_0x^{n-1}+(n-1)a_1x^{n-2}+(n-2)a_2x^{n-3}+\dots).y+ \\ & +\{n(n-1)a_0x^{n-2}+(n-1)(n-2)a_1x^{n-3}+ \\ & +(n-2)(n-3)a_2x^{n-4}+\dots\}.\frac{y^2}{\lfloor 2}+\dots. \end{aligned}$$

Therefore

$$f(x+y) = f(x)+y.f'(x)+\frac{y^2}{\lfloor 2}.f''(x)+\frac{y^3}{\lfloor 3}.f'''(x)+\dots.$$

This is Taylor's theorem.

Now if x be a constant and y variable, when $y = 0$, we get

$$f(x) = f(0);\quad f'(x) = f'(0);\quad f''(x) = f''(0),\ \&c.\ \dots.$$

Consequently,

$$f(y) = f(0)+yf'(0)+\frac{y^2}{\lfloor 2}.f''(0)+\frac{y^3}{\lfloor 3}.f'''(0)+\dots, \qquad (1.58)$$

Maclaurin's theorem, though often referred to as Taylor's, of which it is as we see a particular case. In logic, other than the Hegelian dialectic, $y = y^n$, for all values of n, so that

$$f(y) = f(0)+y.\left\{f'(0)+\frac{f''(0)}{\lfloor 2}+\frac{f'''(0)}{\lfloor 3}+\dots\right\}; \qquad (1.59)$$

and therefore when $y = 1$, the expression in brackets $= f(1) - f(0)$; and we get that $f(y) = f(0) + y\{f(1) - f(0)\}$,

or
$$f(y) = yf(1) + (1-y)f(0), \tag{1.60}$$

giving the law of contradiction and excluded middle.

Hence, unless y is absolutely identical with itself, i.e. $y = y^n$, where n is any number whatsoever from 0 to (∞), it is not true that a thing must be either y or not-y. This is the Hegelian conception.

We must then write, even if $y = y^n$, or values greater than n, but $y \neq y^m$, where $m < n$,

$$f(y) = f(0) + y\left\{f'(0) + \frac{f^n(0)}{\lfloor n}\right\} + y^2\frac{f''(0)}{\lfloor 2} + y^3\frac{f'''(0)}{\lfloor 3} + \dots .$$

Hence

$$f(1) - f(0) - \frac{f''(0)}{\lfloor 2} - \frac{f'''(0)}{\lfloor 3} - \dots = f'(0) + \frac{f^n(0)}{\lfloor n} + \frac{f^{n+1}(0)}{\lfloor n+1} + \dots .$$

Therefore

$$f(y) = yf(1) + (1-y)f(0) - y(1-y^1)\frac{f''(0)}{\lfloor 2} - y(1-y^2)\frac{f'''(0)}{\lfloor 3} - y(1-y^{n-2})\frac{f^{n-1}(0)}{\lfloor n-1} \dots . \tag{1.61}$$

There is thus a continuity between y, y^2, and $y^3, y^4, \dots y^{n-1}$, although $y(1-y^n) = 0$, and the law of contradiction does not hold till $y = y^n$. This is equivalent to the Hegelian dialectic, since by the gradual process of eliminating the higher powers of y we should ultimately attain to the absolute truth which is expressed by $f(y) = y$, $f(1) + (1-y)\, f(0)$; for which both the laws of contradiction and excluded middle hold good, as well as absolute identity (or 'moment') in the very highest degree. In practice therefore truth could never be attained by dialectic, unless we assumed that beyond some certain finite value of n, y becomes absolutely identical with itself, so that further degrees of refinement are unnecessary. This in fact is what Hegel assumes. But it will be shown (p. 226) that no finite degree of refinement would ever enable one to arrive at the truth unless he started with it. It is a mistake, however, to assert, as some do, that the Hegelian logic is entirely erroneous and worthless on this account. By dialectic it is possible only to approximate to the Absolute, but never reach it, whatever Hegel may have thought. Yet where truth has never been attained it is a matter of importance to be able to approximate to it, even if we should never by the method actually arrive at it. So long as

certainty is admitted in perfection, there can be no objection to endeavouring by approximation to attain some measure of the truth in imperfection. It is equivalent to being guided by moral certainty and probability.

We may put

$$y^n = e^{m\theta} = \cos.i.m\theta - i\sin.i.m\theta = (\cos.i.\theta - i.\sin.i\theta)^m *$$

by De Moivre's theorem.

It is obvious therefore how the repeated identity in difference becomes possible, in logical as in trigonometrical analysis. The use of $i^2 = -1$ in Logic is perhaps a novel conception; but there can be no doubt as to its utility, as the sequel will readily show even as to its paramount importance. The use of this mathematical imaginary quantity gives real results in Logic as in mathematics, and need cause no apprehension in one more than in the other, for as we now know its introduction in the theory of relativity has led to unique results.

We may now proceed to the development of $f(x, y)$ on similar lines, and learn to distinguish between those cases in which the laws of contradiction and excluded middle hold good, and when as in approximate truth they do not actually appear to be valid, an assertion passing gradually into its opposite, as in the Hegelian dialectic.

When $x(1-x) = 0$ and $y(1-y) = 0$, then

$$f(xy) = f(1y).x + f(0y).(1-x)$$
$$= x.\{f(11).y + f(10).(1-y)\} + (1-x).\{f(01)y + f(00)(1-y)\}.$$

Therefore we get the general expansion

$$f(xy) = f(11)xy + f(10)x(1-y) + f(01)(1-x)y + f(00).(1-x)(1-y).$$

If, however, $x(1-x) \neq 0$; and $y(1-y) \neq 0$; then (1.62)

$$f(xy) = f(0y) + x.f'(0y) + \frac{x^2}{\lfloor 2} \cdot f''(0y) + \dots .$$

Now $$f(0y) = f(00) + y.f'(00) + \frac{y^2}{\lfloor 2} \cdot f''(00) + \dots,$$

* Certain numbers such as e and π do not satisfy any equation with integral coefficients. They are called transcendental numbers and form a larger group than the cardinal numbers. The 'powers' (see Dantzig, loc. cit.) of an infinite class are called transfinite.

The witty classical scholar who asked 'If a string of terms is equal to nothing, what is the use of talking about it?' a remark that I believe has been erroneously attributed to Dr. Johnson, might have extended his criticism to imaginary roots of unity and even to transfinite numbers, and still more to De Moivre's theorem.

$$f'(oy) = f'(oo)+yf''(oo)+\frac{y^2}{\lfloor 2}f'''(oo)+\ldots,$$

and $$f''(oy) = f''(oo)+yf'''(oo)+\frac{y^2}{\lfloor 2}f^{iv}(oo)+\ldots.$$

We therefore get that

$$f(xy) = f(oo)+yf'(oo)+\frac{y^2}{\lfloor 2}f''(oo)+$$
$$+x\left[f'(oo)+yf''(oo)+\frac{y^2}{\lfloor 2}f'''(oo)+\ldots\right]+$$
$$+\frac{x^2}{\lfloor 2}\left[f''(oo)+yf'''(oo)+\frac{y^2}{\lfloor 2}f^{iv}(oo)+\ldots\right]+$$
$$+\frac{x^3}{\lfloor 3}\left[\ldots\right]+\ldots.$$

Otherwise we may write

$$f(xy) = f(oo)+f'(oo)(x+y)+\frac{f''(oo)}{\lfloor 2}(x^2+2xy+y^2)+$$
$$+\frac{f'''(oo)}{\lfloor 3}(x^3+3x^2y+3xy^2+y^3)+$$
$$+\ldots.$$

In other words

$$f(xy) = f(oo)+f'(oo)(x+y)+\frac{f''(oo)}{\lfloor 2}(x+y)^2+$$
$$+\frac{f'''(oo)}{\lfloor 3}(x+y)^3+\ldots.$$

If $x = x^n$ and $y = y^n$ for any value of n, then

$$f(xy)=A+B(x+y)+Cxy;$$

an expression equivalent to that already obtained; namely

$$f(xy) = \bar{x}\bar{y}+x\bar{y}+\bar{x}y+xy. \qquad (1.63)$$

The fallacy in the Hegelian dialectic, a pitfall into which a classical scholar like Hegel might easily fall, consists in supposing that only two or three syntheses would suffice to reach the truth, whereas the series *might* be infinite according to the nature of the function $f(xy)$. In the expansion

$$f(x) = f(o)+xf'(o)+\frac{x^2}{\lfloor 2}f''(o)+\frac{x^3}{\lfloor 3}f'''(o)+\frac{x^4}{\lfloor 4}f^{iv}(o)+\ldots$$

it is assumed in the dialectic that after a certain value n, where n is a small number, $x = x^n$; or else $f^n(0) = 0$, $f^{n+1}(0) = 0$, &c. In one case it would depend upon the nature of the class x; and in the other upon the nature of the function $f(x)$. In the latter case the laws of contradiction and excluded middle would hold even if that of identity did not beyond the second degree. This is a very curious fact. A similar remark applies to $f(xy)$, or $f(xyz)$, &c. Therefore the possibility of dialectic depends not only upon the nature of the classes x, y, z, ... but upon the nature of the relationship $f(xyz...)$ between these classes. For if the higher differentials of the function $f(xyz...)$ vanish when these vanish, then the process of dialectic is inapplicable. What this means is that whatever degree of identity a class may admit of when considered by itself, in the abstract, that degree might be limited by the relationship that it bears to itself and other classes in the concrete.

The logic of Hegel must be regarded therefore as an approximation to the correct method. Why he should have supposed that by a process of two or three, four or five negations he should have been enabled to reach the Absolute Idea, is by no means clear. The fallacy lies in not recognizing the possibility of the infinite gradation of thought as the relationship of ideas, and of the possibility of an infinite degree of refinement in the nature of the ideas themselves.*

If x, y, z, w, ... be any classes and V_1, V_2, V_3, ... be functions of x, y, z, w, ...; such that $V_1 = 0$; $V_2 = 0$; $V_3 = 0$; ... as the negation of the given premisses. We may put

$$V = V_1 + V_2 + V_3 + \dots .$$

Then if $V = f(x, y, z, w, ...)$ be expanded by Taylor's theorem in the manner shown (p. 80) it is clear that if $V(1-V) = 0$, and $V+(1-V) = 1$, then the terms in V that vanish, i.e. those whose coefficients are $= 0$, will if added together make up the negation of V, i.e. $(1-V)$; and as $V = 0$, the solution of the problem is obtained by adding all such terms whose coefficients in the expansion of V is zero, and putting them equal to unity, i.e. $V = 1$, as containing the truth, the whole truth, and nothing but the truth, being the negation of the negation of the premisses.

Strictly speaking we should take $V^2 = V_1^2 + V_2^2 + V_3^3 + \dots .$ (1.64)

The reason for this is (1) If $V = V_1 + V_2 + V_3 + \dots$ and if we consider any particular term t whose coefficient in V_1 is say At; and in V_2, Bt; in V_3, Ct, &c.; then its coefficient in V is $(A+B+C+\dots)\,t$.

* Of course it *may* be possible to arrive at the truth by a single step, but it usually is a very big jump.

Now unless $A, B, C, \ldots$ are all positive, their sum may vanish, and so not appear at all in V. V therefore may not be logically equivalent to $V_1+V_2+V_3+\ldots$. (2) On the other hand, when we consider the sum of the squares, the coefficients must all be positive, and their sum likewise so. For if $V_1 = a_1t_1+a_2t_2+\ldots$, then since each term t satisfies the law of identity or contradiction, $t_1^2 = t_1$, $t_2^2, = t_2\ldots$, and no two terms are the same, $t_1t_2 = 0$, &c., then $V_1^2 = a_1^2t_1+a_2^2t_2+\ldots$.

Therefore $(A^2+B^2+C^2+\ldots).t$ in this case is positive and V^2 is logically the equivalent of the various propositions V_1, V_2. This is a most important theorem in the Booleian calculus, as its neglect may lead to entirely erroneous results.

The examples given in the sequel, pp. 87–106, will make the practical application of this remarkable theorem quite clear.

Although Boole's method as worked by himself is somewhat complicated, the simplification here proposed renders it easily applicable to most ordinary problems. What is of real interest and importance in his system is the interpretation of the manner that algebraic equations and the ordinary rules of division as well as multiplication yield the right results, even when the process has apparently no logical equivalent in thought. Thus if $x\bar{y} = 0$, and $y\bar{z} = 0$, then we have $(x\bar{y})^2+(y\bar{z})^2 = 0$, or $x\bar{y}+y\bar{z} = 0$, and

$$x = \frac{-y\bar{z}}{\bar{y}};$$

$$\therefore\ x = \frac{0}{0}.yz+\frac{1}{0}.y\bar{z}+\frac{0}{1}.\bar{y}z+\frac{0}{1}.\bar{y}\bar{z}; \qquad (1.65)$$

so that $x = v.yz$; the term v standing for $\frac{0}{0}$; and the factor connected with $\frac{1}{0}$ being $= 0$, the other two terms vanishing from the equation being untrue. How or why the algebraic rules should apply in such a simple manner it is not at first sight easy to apprehend.

The elegance of the method cannot be questioned, but the interpretation of the results is not so easily to be found, unless a closer relationship than is usually imagined between classes in general and those classes denoted by numbers actually exists.

These results follow as before from Maclaurin's theorem, according to which

$$f(x) = f(0)+f'(0).x+f''(0).\frac{x^2}{\underline{2}}+\ldots.$$

But if $$x = x^2 = x^3 \ldots \tag{1.66}$$

$$f(x) = f(0) + x.\left(f'(0) + \frac{f''(0)}{\lfloor 2} + \ldots\right);$$

$\therefore$ if $x = 1$, we get

$$\left(f'(0) + \frac{f''(0)}{\lfloor 2} + \ldots\right) = f(1) - f(0).$$

$$\therefore\ f(x) = f(0) + x.(f(1) + f(0));$$

or $$f(x) = f(1).x + f(0).(1 - x). \tag{1.67}$$

Similarly $$f(xy) = f(1, y).x + f(0, y).\bar{x};$$

$$f(1, y) = f(1, 1).y + f(1, 0).\bar{y};$$

$$f(0, y) = f(0, 1).y + f(0, 0).\bar{y};$$

$\therefore\ f(x, y) = x.\{f(1, 1).y + f(1, 0).\bar{y}\} + \bar{x}\{f(0, 1)y + f(0, 0)\bar{y}\};$

or finally, $f(x, y) = f(1, 1).xy + f(1, 0)x\bar{y} + f(0, 1)\bar{x}y + f(0, 0)\bar{x}\bar{y}.$ (1.68)

In a similar manner $f(x, y, z, \ldots)$ can be expanded, the coefficients being formed by substituting 1 for $x, y, z, \ldots$, and 0 for $\bar{x}, \bar{y}, \bar{z}, \ldots$. A simple illustration is the following. If all x is identical with y which is also $f(x, y)$, then $x = y.f(x, y)$. Therefore

$$f(x, y) = \frac{x}{y} = \frac{1}{1}xy + \frac{1}{0}x\bar{y} + \frac{0}{1}\bar{x}y + \frac{0}{0}\bar{x}\bar{y}.$$

In other words $$f(x, y) = 1.xy + \frac{0}{0} \cdot \bar{x}\bar{y}. \tag{1.69}$$

As in arithmetic the symbol $\frac{0}{0}$ represents an *indefinite class* or number and may be taken to mean *all*, *some*, or *none* of the class to whose expression it is affixed. 1 for ϵ and $\frac{0}{0}$ for $\supset$ in Peano's notation (see p. 67).

The class symbol $\frac{1}{0}$ must represent a class that should be separately equated to 0. This is the algebraic symbol for infinity, and it can be shown that in so far as it does not satisfy the law of non-contradiction it cannot be included in the ordinary expression of logical thought. Boole's proof of this very remarkable fact is based upon the following theorem (*vide* loc. cit., p. 98):

'If a function V, intended to represent any class or collection of

objects, w, be expanded, and if the numerical coefficient, a, of any constituent in its development, do not satisfy the law,

$$a.(1-a) = 0,$$

then the constituent in question must be made equal to 0.'

Proof: Suppose $w = a_1t_1+a_2t_2+a_3t_3+\ldots$ &c.

where t_1, t_2, t_3, &c. represent the constituents, and a_1, a_2, a_3,... &c. the coefficients; and suppose that a_1 and a_2 do not satisfy the law of non-contradiction $a_1(1-a_1) = 0$, and $a_2(1-a_2) = 0$, whilst the other coefficients do so in every case, so that $a_3^2 = a_3$, $a_4^2 = a_4$, &c. Then since $w = V = V^2$, and $t_1^2 = t_1$, $t_2^2 = t_2$, &c., and $t_1t_2 = 0$, &c.

$$w = a_1^2t_1+a_2^2t_2+a_3t_3+a_4t_4+\ldots \text{ &c.}$$

Subtracting the two values of w we get,

$$a_1(1-a_1).t_1+a_2(1-a_2).t_2 = 0;$$

and multiplying this by t_1, we get that $a_1(1-a_1).t_1 = 0$; since $t_1t_2 = 0$, $\therefore$ $t_1 = 0$, and similarly $t_2 = 0$. Thus any coefficient that does not satisfy the fundamental law of non-contradiction cannot enter into the expression of w; and the constituent it accompanies must accordingly vanish. This is obviously true of $\frac{1}{0}$; for, as we have said, the nearer this approaches to infinity the more it departs from the law of non-contradiction.

If $V = f(x, y, z, w,$ &c.) then the expansion of V by Taylor's theorem gives

$$\begin{aligned} V = {} & f(1111)xyzw+f(1110).xyz\bar{w}+ \\ & f(1101)xy\bar{z}w+f(1100)xy\bar{z}\bar{w}+f(1011)x\bar{y}zw+ \\ & f(1010)x\bar{y}z\bar{w}+f(1001)x\bar{y}\bar{z}w+f(1000).x\bar{y}\bar{z}\bar{w}+ \\ & f(0111).\bar{x}yzw+f(0110).\bar{x}yz\bar{w}+f(0101)\bar{x}y\bar{z}w+ \\ & f(0100)\bar{x}y\bar{z}\bar{w}+f(0011)\bar{x}\bar{y}zw+f(0010)\bar{x}\bar{y}\bar{z}w+ \\ & f(0001)\bar{x}\bar{y}z\bar{w}+f(0000).\bar{x}\bar{y}\bar{z}\bar{w}. \end{aligned} \qquad (1.70)$$

The rule therefore is as follows: All the terms whose coefficients vanish are to be collected together and put in logical addition equal to one. The result is a complete solution of the problem. What gives me confidence is that the results come right.

As an illustration we may offer the following example given by Boole, which, however, curiously enough, he does not solve completely, but only partially, although his method really admitted of a general solution, an advantage which we propose to make use of in every case.

(1) Suppose, to take a familiar example, that an analysis of the

properties of a particular class of substances has led to the following general conclusions, viz.:

1st. That wherever the properties A and B are combined, either the property C, or the property D, is present also; but they are not jointly present.

2nd. That wherever the properties B and C are combined, the properties A and D are either both present with them, or both absent.

3rd. That wherever the properties A and B are both absent, the properties C and D are both absent also; and vice versa, wherever the properties C and D are both absent, A and B are both absent also.

Now let us represent the properties A, B, C, and D by x, y, z, and w, respectively.

The three propositions may be written thus:

$$(1)\ x.y = v.(z.\bar{w}+\bar{z}.w);\ (2)\ y.z = v.(x.w+\bar{x}.\bar{w});\ (3)\ \bar{x}.\bar{y} = \bar{z}.\bar{w}; \tag{1.71}$$

in other words, since $z.\bar{w}+\bar{z}.w = 1-(z.w+\bar{z}.\bar{w})$; likewise for x and w:

$$(1)\ x.y.(z.w+\bar{z}.\bar{w}) = 0;\ (2)\ y.z.(x.\bar{w}+\bar{x}.w);\ (3)\ \bar{x}.\bar{y} = \bar{z}.\bar{w}. \tag{1.72}$$

Therefore according to the rules laid down, taking the sum of the squares of (1), (2), and (3), and equating to zero, we obtain

$$V = x.y.(z.w+\bar{z}.\bar{w})^2+y.z.(x.\bar{w}+\bar{x}.w)^2+(\bar{x}.\bar{y}-\bar{z}.\bar{w})^2 = 0.$$

Expanding and substituting 1 for x, y, z, and 0 for $\bar{x}, \bar{y}, \bar{z}, \bar{w}$, we obtain

$$\begin{aligned} V = {} & 1.x.y.z.w+1.x.y.z.\bar{w}+0.x.y.\bar{z}.w+2.x.y.\bar{z}.\bar{w} \\ & +0.x.\bar{y}.z.w+0.x.\bar{y}.z.\bar{w}+0.x.\bar{y}.\bar{z}.w+1.x.\bar{y}.\bar{z}.\bar{w} \\ & +1.\bar{x}.y.z.w+0.\bar{x}.y.z.\bar{w}+0.\bar{x}.y.\bar{z}.w+1.\bar{x}.y.\bar{z}.\bar{w} \\ & +1.\bar{x}.\bar{y}.z.w+1.\bar{x}.\bar{y}.z.\bar{w}+1.\bar{x}.\bar{y}.\bar{z}.w+0.\bar{x}.\bar{y}.\bar{z}.\bar{w} = 0. \end{aligned}$$

Now, as we have seen, the terms whose coefficients do not vanish must be equated to zero; whilst those with zero coefficients are added together and put equal to 1.

We then obtain as the complete solution of the problem the following elegant expression:

$$x.\{y.\bar{z}.w+\bar{y}.(z.w+z.\bar{w}+\bar{z}.w)\}+\bar{x}.\{y.(z.\bar{w}+\bar{z}.w)+\bar{y}.\bar{z}.\bar{w}\} = 1. \tag{1.73}$$

Hence wherever A is present with B, then D must likewise be present and C absent. The presence of C with both A and B is impossible, and inconsistent with the other statements.

Wherever A is present and B absent, then either C or D is present, possibly both.

Wherever A is absent and B present, then either C or D is present separately, but never conjointly, and as appears directly from (3) when both A and B are absent, C and D are likewise so.

The matter can of course be reasoned out and deduced directly, but it would be extremely difficult to think so complicated a matter out fully. The advantage of the algebraic analysis is obvious.

It should be borne in mind, however, that the result just obtained could have been arrived at by Jevons's method of elimination quite simply and the result written down in the manner we have proposed; thus since

$$(1)\, x.y = v.(z.\bar{w}+\bar{z}.w);\ (2)\, y.z = v.(x.w+\bar{x}.\bar{w});\ (3)\, \bar{x}.\bar{y} = \bar{z}.\bar{w}.$$

We have

$$\begin{array}{c}
\cancel{x.y.z.w} \\
\cancel{x.y.z.\bar{w}} \\
x.y.\bar{z}.w \\
\cancel{x.y.\bar{z}.\bar{w}} \\
x.\bar{y}.z.w \\
x.\bar{y}.z.\bar{w} \\
x.\bar{y}.\bar{z}.w \\
\cancel{x.\bar{y}.\bar{z}.\bar{w}} \\
\cancel{\bar{x}.y.z.w} \\
\bar{x}.y.z.\bar{w} \\
\bar{x}.y.\bar{z}.w \\
\cancel{\bar{x}.y.\bar{z}.\bar{w}} \\
\cancel{\bar{x}.\bar{y}.z.w} \\
\cancel{\bar{x}.\bar{y}.z.\bar{w}} \\
\cancel{\bar{x}.\bar{y}.\bar{z}.w} \\
\bar{x}.\bar{y}.\bar{z}.\bar{w}
\end{array} \qquad (1.74)$$

Consequently we obtain

$$x.\{y.\bar{z}.w+\bar{y}.(z.w+z.\bar{w}+\bar{z}.\bar{w})\}+\bar{x}.\{y.(z.\bar{w}+\bar{z}.w)+\bar{y}.\bar{z}.\bar{w}\}=1.$$

This is precisely the same result as that already arrived at by Boole's method, though, as it must be admitted, by a much simpler procedure. It is extremely easy to make mistakes by either method, and those acquainted with both will find it an advantage to check the result in each case by such independent means.

The test of the correct use of Taylor's theorem is in the accuracy of the results when no error is committed in the algebraic process.

(2) Another illustration may be given of the use of these methods. Let us take one from elementary geometry, as given by Boole:

1st. Similar figures consist of all whose corresponding angles are equal and whose corresponding sides are proportional.

2nd. Triangles whose corresponding angles are equal have their corresponding sides proportional, and vice versa.

Let us make $s =$ similar things; $t =$ triangles; $q =$ things having corresponding angles equal; $r =$ things having corresponding sides proportional.

Then (1) $s = q.r$, and (2) $t.q = t.r$. Hence bringing all the terms to one side, squaring, and equating their sum to zero, we obtain

$$(s-q.r)^2+t.(q-r)^2 = 0.$$

In other terms

$$s+q.r-2.s.q.r+t.(q+r-2.q.r) = 0.$$

Substituting 1 for s, t, q, r, and 0 for $\bar{s}$, $\bar{t}$, $\bar{q}$, $\bar{r}$, respectively, we obtain

$$\begin{aligned}
&0.s.t.q.r+2.s.t.q.\bar{r}+2.s.t.\bar{q}.r+1.s.t.\bar{q}.\bar{r}\\
&0.s.\bar{t}.q.r+1.s.\bar{t}.q.\bar{r}+1.s.\bar{t}.\bar{q}.r+1.s.\bar{t}.\bar{q}.\bar{r}\\
&1.\bar{s}.t.q.r+1.\bar{s}.t.q.\bar{r}+1.\bar{s}.t.\bar{q}.r+0.\bar{s}.t.\bar{q}.\bar{r}\\
&1.\bar{s}.\bar{t}.q.r+0.\bar{s}.\bar{t}.q.\bar{r}+0.\bar{s}.\bar{t}.\bar{q}.r+0.\bar{s}.\bar{t}.\bar{q}.\bar{r}
\end{aligned}$$

Taking the terms with zero coefficients and putting them equal to unity, we get

$$s.q.r.(t+t')+\bar{s}.\{t.\bar{q}.\bar{r}+\bar{t}.(q.\bar{r}+\bar{q}.r+\bar{q}.\bar{r})\} = 1.$$

The same result is obtained by Jevons's system of striking out the incompatibles, thus

$s.q.t.r$	~~$\bar{s}.q.t.r$~~
~~$q.t.\bar{r}$~~	~~$q.t.\bar{r}$~~
$q.\bar{t}.r$	$q.\bar{t}.r$
~~$q.\bar{t}.\bar{r}$~~	$q.\bar{t}.\bar{r}$
~~$\bar{q}.t.r$~~	~~$\bar{q}.t.r$~~
~~$\bar{q}.t.\bar{r}$~~	$\bar{q}.t.\bar{r}$
~~$\bar{q}.\bar{t}.r$~~	$\bar{q}.\bar{t}.r$
~~$\bar{q}.\bar{t}.\bar{r}$~~	$\bar{q}.\bar{t}.\bar{r}$

(1.75)

Hence it follows, contrary to what might at first sight have been imagined, that 'Dissimilar figures other than triangles' may have their corresponding angles equal, but the corresponding sides are not proportional, and vice versa.

Jevons's method is no doubt again simpler, but experience shows

that in its application there is a great tendency to overlook or cross out terms by mistake in the midst of so many combinations, so that a check on such a procedure is indispensable. In many cases the simplest expression is obtained by direct solution of the equations without recourse to either of these methods.

(3) An interesting illustration given by Boole from the *Republic* of Plato may be given here, as showing at once the efficacy of our method and its advantage over those of Boole and Jevons, as well as exposing a striking fallacy in Plato's naked and unaided reasoning which Boole overlooks. He assumes in fact an alternative that has been already shown to be by no means justifiable, but which by the analytic method is at once rendered manifest.

Plato thinks he proves that because the Deity is in the best of all possible states He could not change. For since He cannot change for the better, being in the best state, and cannot change for the worse, being ever in the best, He cannot change at all. This reasoning is quite fallacious; since He might change without changing for either a better or a worse state; for though a state cannot be at once both better and worse, it might be neither better nor worse; and remain ever on the same plane whether changing or not. Such a fallacy keeps turning up again and again throughout the whole history of philosophy.

'Either . . . or' might mean the Latin *vel* or *aut . . . aut*. In the latter case they are mutually exclusive conditions; in the former it is only the absence of both that is excluded, so that one or the other separately or both combined is implied. Now in the case we have been considering the presence of both is excluded, but absence of both is not necessarily so by 'either . . . or'; though Plato certainly implied a mutual exclusiveness which was not at all essential, but utterly invalidated his argument.

There is no word for such a disjunctive proposition. The Latin *vel* could be used if applied to the negative terms, thus vel non-a, vel non-b; thereby not excluding the absence of neither a nor b.

We shall give the quotation from the second book of the *Republic* of Plato after Boole, and submit the argument to algebraic analysis; pointing out the fallacy in Plato's reasoning, and correcting that in Boole's.

'Must not that which departs from its proper form be changed either by itself or another thing?'

'Necessarily so.'

'Are not things which are in the best state least changed and disturbed, as the body by meats and drinks, and labours, and every species of plants by heats and winds, and such-like affections? Is not the healthiest and strongest the least changed?'

'Assuredly.'

'And does not any trouble from without least disturb and change that soul which is strongest and wisest? And as to all made vessels, and furnitures, and garments, according to the same principle, are not those which are well wrought, and in a good condition, least changed by time and other accidents?'

'Even so.'

'And whatever is in a right state, either by nature or by art, or by both of these, admits of the smallest change from any other thing.'

'So it seems.'

'But God and things divine are in every sense in the best state.'

'Assuredly.'

'In this way, then, God should least of all bear many forms?'

'Least, indeed, of all.'

'Again, should He transform and change Himself?'

'Manifestly He must do so, if He is changed at all.'

'Changes He then Himself to that which is more good and fair, or to that which is worse and baser?'

'Necessarily to the worse, if He is changed. For never shall we say that God is indigent of beauty or of virtue.'

'You speak most rightly,' said I, 'and the matter being so, seems it to you, O Adimantus, that God or man *willingly* makes himself in any sense worse?'

'Impossible,' said he.

'Impossible then', said I, 'is it that a god should wish to change himself; but ever being fairest and best, each of them ever remains absolutely in the same form.'

The premisses of the argument thus given above are the following:

(1) If the Deity suffers change, He is changed either by Himself or by another.

(2) If He is in the best state, He is not changed by another.

(3) The Deity is in the best state.

(4) If the Deity is changed by Himself, He is changed to a worse state.

(5) If He acts willingly, He is not changed to a worse state.

(6) The Deity acts willingly.

We may express the elements of these propositions as follows, using the same notation as Boole employed, so that the parallelism may the more easily be followed and the differences emphasized.

Let x stand for the proposition 'The Deity suffers change'.

y, 'He is changed by Himself'.

z, 'He is changed by another'.

s, 'He is in the best state'.

t, 'He is changed to a worse'.

w, 'He acts willingly'.

Then the premisses may be written:

$$\left.\begin{array}{l}(1)\ x = v.(y.\bar{z}+\bar{y}.z),\ \text{i.e. } x.(y.z+\bar{y}.\bar{z}) = 0.\\ (2)\ s.z = 0.\\ (3)\ s = 1.\\ (4)\ y.(1-t) = 0.\\ (5)\ w.t = 0.\\ (6)\ w = 1.\end{array}\right\} \quad (1.76)$$

It is quite clear that Jevons's method of writing all the possible combinations of these four unknowns x, y, z, and t (since s and w are equal to 1) is not the simplest method of drawing the legitimate inference from these premisses. Moreover (4) 'If the Deity is changed by Himself, He is changed to a worse state' is, as we have seen, by no means a justifiable assumption, since He might well change without becoming better or worse, both being incompatible with His being in the best state, so that in fact $y.(1-t) \neq 0$.

Assuming, however, (4) to hold good, we have by the method we have all along pursued

$$(x+\bar{x}).(y+\bar{y}).(z+\bar{z}).(w+\bar{w}).(s+\bar{s}).(t+\bar{t}) = 1;$$

since $s = 1$; or $s = 0$; $w = 1$; $w = 0$; $s.z = 0$; $w.t = t$;

we have $(x+\bar{x}).(y+\bar{y}).z.w.s.\bar{t} = 1$.

Now if $y.\bar{t} = 0$, it follows that $x = 0$. From which is left merely $\bar{x}.\bar{y}.\bar{z}.w.s.\bar{t} = 1$ as the only possible conclusion.

If, on the other hand, $y.\bar{t} \neq 0$, then $\{x.y+\bar{x}.(y+\bar{y})\}.\bar{z}.w.s.\bar{t} = 1$; or since obviously $y.\bar{x} = 0$, we have

$$(x.y+\bar{x}.\bar{y}).\bar{z}.w.s.\bar{t} = 1. \quad (1.77)$$

The Deity therefore may or may not suffer change; but if He does, He is changed only by Himself; and in neither case can it be for the worse. According to S. Alexander, the Deity, though representing all that is best at any particular time, yet changes and develops, ever towards a better and better state as the best that is at any time possible and is Himself subject to change and development. This therefore is not only a denial of (4), but a rendering of (3) ambiguous, and indeterminate as a premiss. The conclusion may therefore be written

$$(x.y+\bar{x}.\bar{y}).\bar{z}.w.(s+\bar{s}).\bar{t} = 1. \quad (1.78)$$

Let t_1 stand for the present time. Then since at the present time the Deity is in the best possible state, $t_1 = v.s.t_1$, i.e. $t_1(1-st_1) = 0$ or $t_1.\bar{s} = 0$. Also at any future time He is in the best possible state, hence $\bar{s}.\bar{t}_1 = 0$. So that $s.(t_1+\bar{t}_1) = 1$ or $s = 1$. That is the Deity is always in the best possible state. But as we have seen this does not preclude the possibility of His being changed, and ever to an ever better state; for the best possible may itself be a function

of the time, and as such become better and better. But the best at any particular time is not necessarily the best for all time. And if s_0 stand for the latter, then we should have

$$(x.y+\bar{x}.\bar{y}).\bar{z}.ws.(s_0+\bar{s}_0).\bar{t} = 1, \qquad (1.79)$$

in which t_1 does not enter, and t has nothing to do with the time.

It is quite clear that this method of arriving at a general solution is very much simpler than any of Boole's or Jevons's elaborate expansions. The particular mode of solution must depend upon the circumstances of the case.

(4) The following piece of reasoning for the existence of God, based upon Dr. Samuel Clarke's argument, also taken from Boole, as the succeeding ones are in this section, is particularly worthy of being analysed by these methods. The method of analysis and solution are different.

(1) If motion began in time, the first cause is an intelligent being.

(2) If motion has existed from eternity, either it has been eternally caused by some eternal intelligent being, or it is self-existent, or it must have existed by endless successive communication.

(3) If motion has been eternally caused by an eternal intelligent being, the first cause was an intelligent being.

(4) If it is self-existent, matter is not and never was at rest.

(5) That motion has existed by endless successive communication, and that at the same time it is not self-existent, and has not been eternally caused by some eternal intelligent being, is false.

We may put

x = Motion began in time. $\therefore \bar{x}$ = Motion has existed from eternity.

y = The first cause is an intelligent being.

p = Motion has been eternally caused by some eternal intelligent being.

q = Motion is self-existent.

r = Motion has existed by endless successive communication.

s = Matter is at rest.

Therefore we have

(1) $x = v.y$, or $x.\bar{y} = 0$.

(2) $\bar{x} = v.(p.\bar{q}.\bar{r}+q.\bar{r}.\bar{p}+r.\bar{p}.\bar{q})$, i.e. $\bar{x} = v.(p.\bar{q}+\bar{p}.\bar{q}).\bar{r}$ since $r\bar{p}\bar{q} = 0$ from the proposition (5).

(3) $p = v.y$, or $p.\bar{y} = 0$.

(4) $q.s = 0$.

(5) $r.\bar{p}.\bar{q} = 0$. (1.80)

Now obviously in the first instance we have $s = 0$, or $\bar{s} = 1$, since matter is not at rest.

In general we have

$$(x+\bar{x}).(y+\bar{y}).(p+\bar{p}).(q+\bar{q}).(r+\bar{q})\bar{s} = 1.$$

From (1) and (5) we have

$$x = x.y.\{(p.q+p.\bar{q}+\bar{p}.q).r+(p+\bar{p}).(q+\bar{q}).\bar{r}\}.$$

$$\bar{x} = \bar{x}.\bar{r}.(p.\bar{q}+\bar{p}.q).(y+\bar{y}), \text{ or since } p.\bar{y} = 0 \text{ from (3)},$$

$$\bar{x} = \bar{x}.\bar{r}.\{(p.\bar{q}+\bar{p}.q).y+\bar{p}.q.\bar{y}\}.$$

Now if motion began in time, it obviously could not have existed by endless successive communication (6); $x.r = 0$, i.e.

$$x = x.y.(p+\bar{p}).(q+\bar{q}).\bar{r}.$$

And if motion has existed from eternity, it has existed by endless successive communication; hence (7) $\bar{x}.\bar{r} = 0$.

Hence $\bar{x} = 0$, so that motion has not existed from eternity.

Further, if motion is self-existent, it is and has always been; so it could not have begun in time; i.e. (8) $q.x = 0$, so that

$$x = x.y.\bar{q}.(p+\bar{p}).\bar{r}.\bar{s}.$$

Thus so far although it is quite clear that motion must have had a beginning, and that the first cause was an intelligent being, yet it may not have been the cause of motion. But again, if it has had a beginning, it could not have been eternally caused; so that $x.p = 0$. (9.)

Therefore $$x = x.y.\bar{p}.\bar{q}.\bar{r}.\bar{s}; \qquad (1.81)$$

that is to say that matter is not at rest; motion is not self-existent, nor has it existed by endless successive communications, nor has it been eternally caused by some eternal intelligent being. But motion has had a beginning in time; and the first cause is an intelligent being.

In dealing with the three alternatives in proposition (2) it was assumed that p, q, and r were mutually exclusive; but we may have proceeded from the same premisses, and have arrived at a different conclusion if these three propositions partly overlapped. Obviously the result as regards x would have been the same, but not so as regards $\bar{x}$; for the proposition (2) then assumes the form,

$$\bar{x} = v.\{p.(q+\bar{q}).(r+\bar{r})+q.(r+\bar{r}).(p+\bar{p})+r(p+\bar{p})(q+\bar{q})\}.$$

Now from (7), i.e. $\bar{x}.\bar{r} = 0$, and (5) $r\bar{p}\bar{q} = 0$,

we have $$\bar{x} = v.r.(p.q+p.\bar{q}+\bar{p}.q).$$

Consequently the complete solution is, assuming (6), (8), and (9),

$$\bar{s}.\{x.y.\bar{p}.\bar{q}.\bar{r}+\bar{x}.r.(p.q+p.\bar{q}+\bar{p}.q)(y+\bar{y})\} = 1;$$

or from (3), since $p.\bar{y} = 0$,

$$\bar{s}.\{x.y.\bar{p}.\bar{q}.\bar{r}+\bar{x}.r.(y.(p.q+p.\bar{q}+\bar{p}.q)+\bar{y}.\bar{p}.q).\} = 1.$$

If we assert that either there was not an intelligent first cause or motion is not and was not self-existent, i.e. (10) $y.q = 0$;

$$\bar{s}.\{x.y.\bar{p}.\bar{q}.\bar{r}+\bar{x}.r(y.p.\bar{q}+\bar{y}.\bar{p}.q)\} = 1,$$

so that no definite inference could be drawn as regards an intelligent first cause, except that if there be none such, then motion must be eternal and self-existent, and must have existed by endless successive communication, and could not have been eternally caused by some eternal intelligent being. If, however, motion had existed by endless communication, it would have been eternally caused by some intelligent being, i.e. (11), $r.\bar{p} = 0$; then an intelligent first cause must exist, i.e.

$$\bar{s}.y.\bar{q}.(x.\bar{p}.\bar{r}+\bar{x}.p.r) = 1. \qquad (1.82)$$

It is thus not possible to decide whether motion is eternal or has commenced in time; although the existence of an intelligent cause be admitted.*

(5) Another demonstration given by Clarke is the following:

(1) Something is.

(2) If something is, either something always was, or the things that now are have arisen out of nothing.

(3) If something is, either it exists in the necessity of its own nature, or it exists by the will of another being.

(4) If it exists in the necessity of its own nature, something always was.

(5) If it exists by the will of another being, then the hypothesis, that the things which now are have arisen out of nothing, is false.

Let x = Something is.

y = Something always was.

z = The things which now are, have arisen out of nothing.

p = It exists in the necessity of its own nature (i.e. the something spoken of above).

q = It exists by the will of another Being.

Then we have

(1) $x = 1$.

* Boole's equations are extremely difficult to manipulate, and Jevons's method too mechanical. The method I have adopted is that of the ordinary reasoning process, powerfully aided by the use of symbols.

(2) $x = v.(y.\bar{z}+\bar{y}.z)$; since $y.z = 0$; obviously this term need not be included in the alternative between x and y.

(3) $x = v.(p.\bar{q}+\bar{p}.q+p.q)$; since here $p.q$ is not necessarily $= 0$. (1.821)

(4) $p.\bar{y} = 0$.

(5) $q.\bar{z} = 0$.

$$(x+\bar{x}).y+\bar{y}).(z+\bar{z}).(p+\bar{p}).(q+\bar{q}) = 1;$$

whence from (1, 2, and 3)

$$x.(y.\bar{z}+\bar{y}.z).(p.\bar{q}+\bar{p}.q+p.q) = 1,$$

and from (4 and 5) this leads to

$$x.y.\bar{z}.(p.\bar{q}+\bar{p}.q+p.q) = 1. \qquad (1.822)$$

The complete solution therefore is the following:

Something is, and something always was, and the things which now are, have not arisen from nothing.

It is impossible, however, to decide from the premisses whether what exists, exists in the necessity of its own nature, or by the will of another being; and even in the former case also by the will of that being.

(6) Another illustration of a somewhat similar nature is the following:

(1) Something has always existed.

(2) If something has always existed, either there has existed some one unchangeable and independent being, or the whole of existing things has been comprehended in a succession of changeable and dependent beings.

(3) If the Universe has consisted of a succession of changeable and dependent beings, either that series has had a cause from without, or it has had a cause from within.

Let x = Something has always existed.

y = There has existed some one unchangeable and independent being.

z = There has existed a succession of changeable and dependent beings.

p = That series had a cause from without.

q = That series has had a cause from within.

Then

$$\left.\begin{array}{l}(1)\ x = 1.\\ (2)\ x = v.(y.\bar{z}+\bar{y}.z).\\ (3)\ z = v.(p.\bar{q}+p.\bar{q}).\end{array}\right\} \qquad (1.823)$$

Hence $x.(y.\bar{z}+\bar{y}.z).(p+\bar{p}).(q+\bar{q}) = 1.$

(4) If the series has had a cause from without, there has existed an independent being, i.e. $p\bar{y} = 0$.

(5) If the series has had a cause from within, there has existed a succession of changeable and dependent beings, i.e. $q.\bar{z} = 0$.

Therefore $x.(y.\bar{z}.(p+\bar{p}).\bar{q}+\bar{y}.z.\bar{p}.q) = 1.$ (1.824)

Now Clarke uses the two following premisses which previous to Kant might have been accepted without question or criticism by most philosophers. What he asserts is: (*a*) x has not had a cause from without (because it includes, by hypothesis, all things that exist); (*b*) it has not had a cause from within (because no part is necessary, and if no part is necessary, the whole cannot be necessary).

Hence $p = 0$, and $q = 0$; so that we have

$$x.y.\bar{z}.\bar{p}.\bar{q} = 1.$$

Thus the cause is neither from without nor within; but that there has existed some one unchangeable and independent being is indisputable.

But when (3) and (4) and (5) are assumed instead of (*a*) and (*b*), it follows that there cannot have been a cause both from without and from within; but either separately or neither is possible. If there is no cause from without, i.e. $p = 0$, there may or may not be a cause from within; and the existence of a first cause is not an inevitable consequence from the premisses; but if there is no cause from within, i.e. $q = 0$, then it does follow that a first cause did give rise to the series of events. Yet although there must have existed in that case some one unchangeable and independent being, the series may or may not have had a cause from without.

This is of great importance in dealing with the Kantian antinomies.

(7) The following argument for the self-existence of an unchangeable and independent Being is of particular value in its analytical bearing upon this system.

The premisses are:

(1) Every being must either have come into existence out of nothing, or it must have been produced by some external cause, or it must be self-existent.

(2) No being has come into existence out of nothing.

(3) The unchangeable and independent Being has not been produced by an external cause.

Let x = Beings which have arisen out of nothing.

y = Beings which have been produced by an external cause.

z = Beings which are self-existent.

w = The unchangeable and independent being.

Then (*a*) $x.\bar{y}.\bar{z}+y.\bar{z}.\bar{x}+z.\bar{x}.\bar{y} = 1$; if *aut—aut*.

But $x = 0$. Therefore $\bar{x}(y.\bar{z}+\bar{y}.z).(w+\bar{w}) = 1$.

And since $w.y = 0$, we get that

$$\bar{x}[y.\bar{z}.\bar{w}+\bar{y}z(w+\bar{w})] = 1;$$

otherwise, $$\bar{x}.(w.\bar{y}.z+\bar{w}.(y.\bar{z}+\bar{y}.z)) = 1.$$

Hence so far as the premisses go, there may be self-existent beings, other than the unchangeable and independent being, that are not produced by an external cause.

On the other hand, x, y, z may not be mutually exclusive in the disjunctive proposition *vel*, which may be expressed $\bar{x}.\bar{y}.\bar{z} = 0$.

The solution then assumes the form

(*b*) $$\bar{x}.\{w.\bar{y}.z+\bar{w}.(y.z+y.\bar{z}+\bar{y}.z)\} = 1. \qquad (1.83)$$

Hence, as before, the unchangeable and independent being, like everything else, did not arise out of nothing; but it is itself self-existent and not produced by an external cause. Yet other self-existent beings are also possible, although unlike the preceding case they might, though self-existent, have had their origin in an external cause; whereas in the previous case the assumed exclusiveness of the three alternatives required that if a being be self-existent it could not have been produced by an external cause. It may be self-existent after it has been produced.

(8) The following example of simplified algebraic solution may also be given. The proposition is that 'The Self-Existent Being must of necessity be infinite and omnipresent'. The premisses are the following:

(1) If a finite being is self-existent, it is a contradiction to suppose it not to exist.

(2) A finite being may without contradiction be absent from one place.

(3) That which may without contradiction be absent from one place, may without contradiction be absent from all places.

(4) That which may without contradiction be absent from all places, may without contradiction be supposed not to exist.

We may put

x = Finite beings.

y = Things that are self-existent.

z = Things which it is a contradiction to suppose not to exist.

w = Things which may be absent without contradiction from one place.

t = Things which without contradiction may be absent from every place. Then we have the premisses as follows:

$$\left.\begin{array}{ll}(1) & x.y.\bar{z} = 0.\\ (2) & x.\bar{w} = 0.\\ (3) & w.\bar{t} = 0.\\ (4) & t.z = 0.\end{array}\right\} \qquad (1.831)$$

Now from (2) and (3) we obviously get that $x.\bar{t} = 0$. Therefore we get for x the value $x = x.w.t.\bar{z}(y+\bar{y})$; but the first term must obviously vanish, since from (1) $x.y.\bar{z} = 0$. Therefore $x = x.\bar{y}.\bar{z}.t.w$.

Also from (3) and (4) we have

$$\bar{x} = \bar{x}.(y+\bar{y}).\{t.\bar{z}+\bar{t}.(z+\bar{z})\}.\{t.(w+\bar{w})+\bar{t}.\bar{w})\} = 1.$$

The terms within the two large brackets become

$$\{t.\bar{z}.(w+\bar{w})+\bar{t}.\bar{w}.(z+\bar{z})\}.$$

Hence the complete solution of the problem is given by the expression

$$x.\bar{y}.\bar{z}.w.t+\bar{x}.(y+\bar{y}).(t.\bar{z}.(w+\bar{w})+\bar{t}.\bar{w}.(z+\bar{z})) = 1. \quad (1.84)$$

Thus 'No finite being is self-existent' and 'All self-existent beings are infinite', or 'Whatever is self-existent is infinite'.

The beauty and simplicity of this method as compared with either Boole's or Jevons's is apparent as the reasons can be followed. Obviously with skill, much labour may be avoided in the reduction of terms; and a little practice will very much serve towards perfection in this respect.

(9) It can now be shown that there cannot be more than one necessary and independent being; for (1) If there are two or more necessary and independent beings, either of them may be supposed to exist alone. (2) If either can be supposed to exist alone, it is not a contradiction to suppose the other not to exist. (3) If it is not a contradiction to suppose this, there are not two necessary and independent beings; for if there were two such, we could not suppose them as necessary beings not to exist.

Briefly put,

x = 'There exist two necessary and independent beings.'

y = 'Either may be supposed to exist alone.'

z = 'It is not a contradiction to suppose the other not to exist.'

Hence (1) $x.\bar{y} = 0$; (2) $y.\bar{z} = 0$; (3) $z.x = 0$.

Now $(x+\bar{x}).(y+\bar{y}).(z+\bar{z}) = 1.$

Therefore $x.y.z.\bar{x}+\bar{x}.(y.z+\bar{y}.(z+\bar{z})) = 1,$

or since it follows that $x = 0$, we get the complete solution

$$\bar{x}.(y.z+\bar{y}.(z+\bar{z})) = 1. \qquad (1.85)$$

Thus there cannot be two necessary and independent beings.

The interpretation of the term $\bar{y}.(z+\bar{z})$ may be of special interest as showing the exactness of the reasoning, namely that 'Although there cannot exist two necessary and independent beings, it may or may not be true that either may be supposed to exist alone, and if the latter is not the case then, it may or may not be a contradiction to suppose the other not to exist'.

Bishop Butler in criticizing this argument says: 'I do not conceive that the idea of ubiquity is contained in the idea of self-existence, or that it directly follows from it, any otherwise than that whatever exists must exist *somewhere*', in other words, 'necessary existence implies existence in some parts of space, though not necessarily in every part'. Clarke clearly does not prove omnipresence as a necessary attribute of self-existence.

(10) An important application of the algebraic method, in which, however, I have detected an error in one of Boole's inferences, is given in his analysis of Clarke's arguments for the origin of evil, the *Πόθεν τὸ κακόν* of Greek philosophy. Briefly it runs as follows:

'*All that we call evil* is *either an evil of imperfection*, as the want of certain faculties or excellencies which other creatures have; or *natural evil*, as pain, death, and the like; or *moral evil*, as all kinds of vice. The *first of these is not properly an evil*; for every power, faculty, or perfection, which any creature enjoys, being the free gift of God . . . it is plain that the want of a certain faculty or perfection in any kind of creatures, which never belonged to their natures is no more an evil to them, than their never having been created or brought into being at all could properly have been called an evil. The *second kind of evil*, which we call *natural evil*, is either a *necessary consequence of the former*, as death to a creature on whose nature immortality was never conferred; and it is no more properly an evil than the former. Or else it is *counterpoised on the whole with as great or greater good*, as the afflictions and sufferings of good men, and then also it is not properly an evil; or else, lastly, it is *a punishment, and then it is a necessary consequence of the third and last kind of evil*, viz., *moral evil*. And this arises wholly from the abuse of liberty which God gave to His creatures for other purposes, and which it was reasonable and fit to give them for the perfection of the whole creation. Only they, contrary to God's intention and command, have abused what was necessary to the perfection of the whole, to the corruption and deprivation of themselves. And thus all sorts of evil have entered into the world without any diminution to the infinite goodness of the Creator and Governor thereof.'

The words which I have italicized form the elements for the analysis

of the argument. The reasoning takes two lines of discussion, according as the three kinds of evil, and the consequences, and compensations referred to are regarded as mutually exclusive *aut—aut* or as classes of ideas, capable of overlapping *vel.*

Using Boole's notation, for simplicity, in order the more easily to compare the parallel methods of procedure, we may put:

w = reputed evil.
x = evil of imperfection.
y =natural evil.
z = moral evil.
t = absolute evil.
p = consequence of evil of imperfection.
q = things compensated with greater good.
r = consequence of moral evil.

There are thus eight terms to be dealt with, a rather difficult and extremely trying order for Jevons's method of cancelling incompatible combinations, as there are 2^8 or 256 combinations to start with, and the precise determination of those to be eliminated is by no means obvious. The premisses are:

$$\left.\begin{array}{l}(1)\ w = v.(x.\bar{y}.\bar{z}+y.\bar{z}.\bar{x}+z.\bar{x}.\bar{y}).\\ (2)\ x = v.(1-t).\\ (3)\ y = v.(p.\bar{q}.\bar{r}+q.\bar{r}.\bar{p}+r.\bar{p}.\bar{q}).\\ (4)\ p.\bar{q}+\bar{p}.q = v.(1-t).\\ (5)\ t = v.w.\end{array}\right\} \quad (1.86)$$

Otherwise these may be written:

$$\left.\begin{array}{l}(1)\ w.(1-x\bar{y}\bar{z}-y\bar{z}\bar{x}-z\bar{x}\bar{y}) = 0.\\ (2)\ x.t = 0.\\ (3)\ y.(1-p\bar{q}\bar{r}-q\bar{r}\bar{p}-r\bar{p}\bar{q}) = 0.\\ (4)\ (p.\bar{q}+\bar{p}.q).t = 0.\\ (5)\ t.\bar{w} = 0.\end{array}\right\}$$

Now from (1), (2), (4), and (5) it is quite easy to see that a relation for t follows thus: $t = t.\bar{x}.w.(y\bar{z}+\bar{y}z).(pq+\bar{p}\bar{q}).(r+\bar{r})$, which further simplifies from (3) to

$$t = t.\bar{x}.w.[y\bar{z}\bar{p}\bar{q}r+\bar{y}z.(pq+\bar{p}\bar{q})\,(r+\bar{r})]. \quad (1.861)$$

The value for $\bar{t}$ is not quite so simple, although its determination is of importance for the complete solution of the problem, particularly for the determination at a glance of the values of the other terms.

$$\bar{t} = \bar{t}.w(y\bar{z}\bar{x}\,((p\bar{q}+\bar{p}q)\bar{r}+r\bar{p}\bar{q})+\bar{y}(x\bar{z}+\bar{x}z).(p+\bar{p})\,(q+\bar{q})\,(r+\bar{r}))$$

$$+\bar{t}.\bar{w}\,[y\,((p\bar{q}+\bar{p}q)(\bar{r}+r\bar{p}\bar{q})+\bar{y}(p+\bar{p})\,(q+\bar{q})\;\;(r+\bar{r})](x+\bar{x}).$$
$$(z+\bar{z}). \qquad (1.87)$$

And of course $t+\bar{t}=1$.

Now Boole has an ingenious method of eliminating as many of the unknowns as may not be required so as to obtain a relationship amongst those that remain. This method of elimination is useful in some cases. There are, however, many dangers of errors creeping in through such more or less mechanical methods. Thus in reply to the question 'Is any relation implied between absolute evils, evils of imperfection, and consequences of evils of imperfection?' i.e. t, x, and p, he concludes that $t=v.\bar{x}.\bar{p}$; but this inference is obviously wrong, since it is quite clear from our argument directly from the premisses, without any elaborate calculations by any 'transcendental' rules, that $t=v.\bar{x}.(pq+\bar{p}.\bar{q})$.

Absolute evil though not an evil of imperfection may nevertheless be a consequence of evil of imperfection; and in such a case it is compensated with greater good. It follows further that, in such a case, absolute evil is a moral evil and not a natural one, and that it may or may not in that case be a consequence of moral evil.

$$t=v\bar{x}w\bar{y}z(pq+\bar{p}\bar{q})\,(r+\bar{r}). \qquad (1.88)$$

There is a tacit assumption made in this argument that is brought out very clearly by this method of viewing the subject as a whole. If every evil of imperfection, every natural evil, and all moral evil be a reputed evil, as is obviously implied, then $x\bar{w}=0$; $y\bar{w}=0$; $z\bar{w}=0$. This can, as it appears, make no difference to the value of t or absolute evil; but

$$\bar{t}=v.\bar{w}.\bar{x}.\bar{y}.\bar{z}.(p+\bar{p})\,(q+\bar{q})\,(r+\bar{r})+vw. \qquad (1.89)$$

Hence what is not reputed an evil at all may or may not be a consequence of evil of imperfection, or of moral evil; and it may or may not be compensated with greater good. Whereas if those three propositions are not definitely stated, then various cases of evil, whether of moral, natural, or of imperfection, that are not reputed evil at all may be found; with particular limitations in the case of natural, such evils are not reputed as evils. We need not, however, dwell upon these.

To complete the argument as regards the abuse of liberty. Let us put $u=$ consequence of abuse of liberty. Then we have obviously

(6) $z.\bar{u}=0$. All moral evil is a consequence of the abuse of liberty.

(7) $r.\bar{u}=0$. Consequences of moral evil are consequences of the abuse of liberty. Hence absolute evil must be a consequence of the abuse of liberty, since from (1.861) it must be either moral

evil (z) or a consequence of moral evil (r). We should therefore insert u in the value for t. In the value $\bar{t}$ the factor $(u+\bar{u})$ must be added with the equations $z\bar{u} = 0$ and $r\bar{u} = 0$. These obviously make no difference in the second expression for $\bar{t}$.

Let us now consider the same argument with the particles *either, or* as not absolutely disjunctive *vel*. The premiss (4) may also be expressed as $p.t = 0$ and $q.t = 0$, because $p\bar{q}+\bar{p}q+pq = v\bar{t}$. $\therefore (1-\bar{p}\bar{q})t = 0$ or $t = t\bar{p}\bar{q}$. $\therefore\ tp = 0$: $tq = 0$. We then easily get that

$$t = t.\bar{x}.w.\bar{p}.\bar{q}.u.\{y.(z+\bar{z}).r+\bar{y}.z.(r+\bar{r})\}. \qquad (1.90)$$

The expression for $\bar{t}$ is far too complicated, though the procedure is somewhat the same as before.

Here of course $t = v.\bar{x}.\bar{p}$.

The premisses are

$$\left.\begin{array}{l} (1)\ w.\bar{x}.\bar{y}.\bar{z} = 0. \\ (2)\ x.t = 0. \\ (3)\ y.\bar{p}.\bar{q}.\bar{r} = 0. \\ (4)\ p.t = 0,\ q.t = 0. \\ (5)\ t.\bar{w} = 0. \\ (6)\ z.\bar{u} = 0. \\ (7)\ r.\bar{u} = 0. \end{array}\right\} \qquad (1.91)$$

Boole's method of elimination gives the following results.

The plan of elimination is this. Suppose that we wish to eliminate w, between (1) and (5). We add the squares of the terms in the two equations together and put w equal to 1 and 0, respectively; and put the algebraic product of the two equal to zero. This gives us the required relationship between the remaining terms. Thus we get $t.\bar{x}.\bar{y}.\bar{z} = 0$. Now combining this in a similar way with (3), we obtain $t.\bar{x}.\bar{z}.\bar{p}.\bar{q}.\bar{r} = 0$, and from (4) $t.\bar{x}.\bar{z}.\bar{r} = 0$. By the aid of (6) we have $t.\bar{x}.\bar{r}.\bar{u} = 0$, and from (7), $t.\bar{x}.\bar{u} = 0$, and from (2) it follows that finally $t.\bar{u} = 0$.

The following inferences, amongst others, may also be drawn by this very simple method of elimination. $\bar{t}.\bar{u}.\bar{r} = 0$; $t.\bar{z}.\bar{r} = 0$; $t.\bar{y}.\bar{z} = 0$, &c.

In the first case where the particles *either, or* are absolutely disjunctive, the elimination becomes more difficult, as the equations are more complicated. Thus from (1), (2), and (5), p. 102, we obtain that $t.\{1-\bar{x}.(y\bar{z}+\bar{y}z)\} = 0$ or eliminating y, between this and (3), we get $\{(1-\bar{x}\bar{z})+(1-r\bar{p}\bar{q})\}.t.(1-\bar{x}z) = 0$, on account of (4), and eliminating q and r, we get

$$t.(1-\bar{x}z).(2-\bar{x}\bar{z}).(2-\bar{x}\bar{z}-\bar{p}) = 0.$$

From which it follows that

$$t = \frac{0}{(1-\bar{x}z).(2-\bar{x}\bar{z}).(2-\bar{x}\bar{z}-\bar{p})} \tag{1.92}$$

or $$t = \frac{0}{4}.xzp + \frac{0}{2}.xz\bar{p} + \frac{0}{4}.x\bar{z}p + \frac{0}{2}.x\bar{z}\bar{p} +$$

$$+ \frac{0}{0}.\bar{x}zp + \frac{0}{0}.\bar{x}z\bar{p} + .\frac{0}{1}\bar{x}\bar{z}p + \frac{0}{0}.\bar{x}\bar{z}\bar{p};$$

otherwise $t = v.\bar{x}.\{z(p+\bar{p})+\bar{z}\bar{p}\}$.

This solution is clearly correct, as it is in accordance with our previous results, and contrary to that given by Boole, who gets (p. 222, loc. cit.) $t = \frac{0}{(p+x)} = \frac{0}{0}.(1-p).(1-x)$, which is obviously wrong though true when the disjunctive is *vel*. It shows how easily one might go astray in the manipulation of these symbols; and how desirable it is accordingly that the results in every case should be verified by entirely independent means.

With a little practice this method of elimination can be employed with great facility.

It will be observed that $\frac{0}{0}$ is not an arithmetical fraction, but nevertheless a *relation*.

The utility of Boole's particular methods of expansion and elimination can be fully displayed only in connexion with the mathematical theory of probabilities. The following familiar example, however, illustrates its extraordinary power.

(11) Let the observation of a class of natural productions be supposed to have led to the following results:

(1) That in whichsoever of these productions the properties A and C are missing, the property E is found, together with one of the properties B or D, but not with both.

(2) That wherever the properties A and D are found while E is missing, the properties B and C will either both be found, or both be missing.

(3) That wherever the property A is found in conjunction with either B or E, or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E, or both of them.

Putting x, y, z, w, and v for the presence of the properties A, B, C, D, and E, respectively, the premisses are:

$$\left.\begin{array}{l}(1)\ \bar{x}\bar{z} = q.v(y\bar{w}+\bar{y}w).\\(2)\ \bar{v}xw = q.(yz+\bar{y}\bar{z}).\\(3)\ xy+xv\bar{y} = w\bar{z}+\bar{w}z.\end{array}\right\} \quad (1.93)$$

The solution of these equations may be obtained quite easily either by Jevons's method of elimination of all combinations that do not tally with the three given equations; or by the following method adopted by Boole. Transpose all the three equations to one side, and we obtain,

$$\left.\begin{array}{l}(1)\ \bar{x}\bar{z}\{1-v.(y\bar{w}+\bar{y}w)\} = 0.\\(2)\ \bar{v}wx.(y\bar{z}+\bar{v}z) = 0.\\(3)\ x(y+\bar{y}v)-(z\bar{w}+\bar{z}w) = 0.\end{array}\right\} \quad (1.94)$$

If we take the sum of the squares of these and put them equal to V we get

$$V = \bar{x}\bar{z}\{1-v(y\bar{w}+\bar{y}w)\}^2+\bar{v}xw(y\bar{z}+\bar{v}z)^2+\{x(y+\bar{y}v)-(z\bar{w}+\bar{z}w)\}^2.$$

If we put x, y, z, w, and v, equal to 1 and 0, respectively, and collect the terms for which $V = 0$, we obtain

$$x\{y(z\bar{w}(v+\bar{v})+\bar{z}wv)+\bar{y}((z\bar{w}+\bar{z}w)v+\bar{z}\bar{w}\bar{v})\}+$$
$$+\bar{x}\{y(zw(v+\bar{v})+\bar{z}\bar{w}v)+\bar{y}zw(v+\bar{v})\} = 1. \quad (1.95)$$

The complete solution of the problem then is:

If A is present, B may or may not be present; but if it is, then C or D is also present separately, but never conjointly. When C is then present with B and A, E may or may not be present; but if in the alternative D be present with B and A, then E must be present. And when A is present without B, then E may or may not be present; but if it is then C or D is also present separately and never conjointly; whereas if E and B be both absent, then both C and D are likewise absent.

If A is absent and B present, then C and D are either conjointly present or they are both absent. In the former case E may or may not be present; whereas in the latter case E must be present. If A and B are both absent then C and D are always present together, though E may or may not be found with them. This includes every possible case.

The non-mathematical student can always arrive at a solution by Jevons's method. But the mathematician has the advantage of being able to check his results by different means.*

* The Booleian method is given not only to demonstrate its accuracy in general, but as an aid to the subsequent discussion as to the interpretation of the coefficients of the terms and true significance of the system, rather than as a commendation of its utility in this particular case.

CHAPTER IV

EXISTENTIAL PROPOSITIONS

§ 1

WHEN two members of the class x differ only in regard to the attribute y, then $x = x(y+\bar{y})$. But if $y = 0$, then $x = x\bar{y}$; or since then $\bar{y} = 1$, $x = x$. If therefore x differs further from x it can only be, as we have seen, on account of differences in the powers of x, where $x = x^n$; and this is possible if x has the value of an imaginary root of unity, when n is greater than 3. In such cases $x.\bar{x}$ is not necessarily $= 0$. But there is no way in which we can conceive such a state of affairs. Only experience and the sense of continuity render this possible. Algebraically it is quite obviously so; though unintelligible to ordinary common sense. The illustrations we have given afford an explanation only when the two x's are not actually identical in thought. In the algebraical equation they are identical, and yet different. The conception $t.\sqrt{-1}$ in the theory of relativity is precisely of this nature. The square of the time is real, but the time is not so, as long as $\sqrt{-1}$ enters into the calculation. But the algebraic treatment leads to right results, in accordance with experience which confirms them. Likewise in thought generally transcended by the algebra of logic, the differences in identity are surmounted. 'Sapientia naturam superavit.'

Only if logic be regarded as a branch of mathematics is the paradox rendered in any way intelligible; though not translatable into the language of common thought. To think mathematically is to recognize the function that the part plays in the whole.

[It is obvious that the contention so often put forward, and repeated as recently as January 1928 by Mr. H. W. B. Joseph in the *Journal of Philosophical Studies*, that Logic is not and cannot be regarded as a branch of Mathematics is at variance with my view. It is a reiteration of the old fallacy that the latter deals only with numbers but that Logic, which is the science of classification, does not. Inference and judgement are but modes of classification. And the problems of the logician when analysed are found to be particular cases of the more general problems presented to the mathematician.]

If we suppose that no statement is strictly speaking true, then all possible combinations of terms may be held to be possible propositions. For example, in the case of two classes x and y; there are the combinations $(x+\bar{x}).(y+\bar{y}) = 1$, which on

expansion give $(xy+\bar{x}y+x\bar{y}+\bar{x}\bar{y})=1$. Now unless some one term vanishes as a general proposition, all four classes must be held to be possible. We should therefore have x's that are y's and y's that are x's. Also x's that are not y's and y's that are not x's; as well as those that are neither x's nor y's.

In general if there be n terms $x_1x_2x_3...x_n$, the number of combinations is 2^n-1, when the possibility of none of them is excluded. But logically this should be formally regarded as part of the universe of discourse; so that the total number of classes is 2^n, as the logical method of classification shows, thus:

$$(x_1+\bar{x}_1)(x_2+\bar{x}_2)(x_3+\bar{x}_3)\dots(x_n+\bar{x}_n)=1. \qquad (1.96)$$

The same result follows from the binomial theorem, for

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2!}.a^{n-2}b^2+\frac{n(n-1)(n-2)}{3!}\dots.$$

Now the coefficients of a^nb^r are the expressions for the number of combinations of n things taken r at a time. Hence if we put $a=b=1$, we get

$$2^n=1+n+\frac{n(n-1)}{2!}+\frac{n(n-2)(n-3)}{3!}+\dots. \qquad (1.97)$$

Therefore the total number of combinations is $=2^n-1$. With the class $\bar{x}_1\bar{x}_2\bar{x}_3...\bar{x}_n$ this becomes 2^n.

When the full expansion of the logical function

$$(x_1+\bar{x}_1)\dots(x_n+\bar{x}_n)=1$$

is taken into account all the subtle distinctions made by such logicians can be included so as to cope with every shade of meaning. The $x\bar{y}=0$ may really be of the nature $x_1x_2...x_n\bar{y}_1\bar{y}_2...\bar{y}_n=0$. When Russell and Peano tell us that the older philosophers were mistaken in supposing that two propositions 'this is yellow' and 'buttercups are yellow' are not of the same type, and that it is not true to affirm that all propositions are of this type, they state what is no doubt true and most important; but the notation invented by Peano and adopted by Russell and Whitehead, though admirably suited for their purpose, does not appear to be applicable to the analysis of all problems in a general manner. The chief objection raised to the method here employed is that it apparently does not include the explanation of order. The combination of the two methods obviously is the proper course; for as the larger includes the smaller, so does the Booleian system include the Peanonian, as all propositions in which order is of importance must ultimately be included in the more general logic of the relation of

classes irrespectively of the order in the sub-classes. Different symbols must be employed where the order is reversed, thus $p = xy$, $q = yx$. The Peanonian notation is of the nature of a simplified language, rather than a calculus by which calculations or abstruse reasoning may be performed. The latter indeed is admirable for the analysis of definitions; and almost the whole of Burali-Forti's *Logica Matematica* is confined to the process of translating propositions into this simplified notation. In the matter of elaborate reasoning it is of the nature of grammar, and it is in fact of practical value as such.

§ 2

According to Peano the proposition 'x is y' may be written in the form $x \in y$; by which it is understood that x is a member of the class y; whilst $x \supset y$ indicates that x is contained in y: two different forms apparently of the same proposition 'x is y'. The former according to Boole would be expressed by $x = vy$, and the latter by $x = iy$, i.e. in the first case x is some thing or things that are y; and in the second one x is a y. The term 'exists', however, has more than one meaning, and therefore it is held by Peano that such an expression is inadequate. But in the Booleian calculus whatever shades of refinement of meaning may be desired can always be expressed by the introduction of the terms $(e_1+\bar{e}_1)\ldots(e_n+\bar{e}_n)$ implied or understood. Burali-Forti remarks that 'coloro che pretendano identificare $\in$ a $\supset$ possono soltanto formare una *logica infantile*'.

According to Boole if $x = vy$, then $x\bar{y} = vy\bar{y} = 0$. And conversely if $x\bar{y} = 0$, then $x = \frac{0}{y} = \frac{0}{0}\cdot y + \frac{0}{1}\cdot\bar{y}$.

If, however, the proposition 'x is y' is true only when the copula 'is' is used in a certain particular sense E, then $x\bar{y}E = 0$; that is x is y subject to the condition E, so that $x\bar{y} \neq 0$ unless the factor E is included or

$$x\bar{y}E = 0; \quad x = \frac{0}{\bar{y}E} = \frac{0}{0}\cdot yE + \frac{0}{0}y\bar{E} + \frac{0}{1}\cdot\bar{y}E + \frac{0}{0}\cdot\bar{y}\bar{E}^{*}$$

$$= \frac{0}{0}(yE + y\bar{E} + \bar{y}\bar{E}). \qquad (1.98)$$

It may be said that the proposition 'x is y' does in fact admit of several meanings; and in that case the equivalent forms hold only when the terms implied or understood also admit of it. This clearly applies to all propositions whatsoever; so that quibbling

* If E holds then x is y, but otherwise x may or may not be y.

about words is not permissible. If a shade of meaning is required then it must be introduced. We can go on for ever refining as Hegel does, ever drawing finer and finer distinctions, until we get tired of it or else are convinced that further distinction becomes self-contradictory. It is extremely difficult to make sure that there is not some condition implied under which alone a proposition holds. The solution given in (1.98) from $x\bar{y}E = 0$ means that if the condition E holds then x is y. But not that if x is y, E holds. This would be expressed by

$$xy\bar{E} = 0; \quad \text{or } x = \frac{0}{y\bar{E}} = \frac{0}{0}.(yE+\bar{y}E+\bar{y}\bar{E}). \qquad (1.99)$$

In one case if x is y, E may or may not hold; but if E does not hold x is not y. In the other case if x is y, E must hold; but if it is not y, then E may or may not hold.

The equations (1) $x\bar{y}E = 0$, and (2) $xy\bar{E} = 0$, can be solved as follows, for E

$$(1) \qquad E = \frac{0}{x\bar{y}} = \frac{0}{0}.xy+\frac{0}{1}.x\bar{y}+\frac{0}{0}.\bar{x}y+\frac{0}{0}\bar{x}\bar{y}; \qquad (2.00)$$

and from

$$(2)\ xy(1-E) = 0; \text{ or } E = \frac{xy}{xy} = \frac{1}{1}.xy+\frac{0}{0}.x\bar{y}+\frac{0}{0}.\bar{x}y+\frac{0}{0}.\bar{x}\bar{y}. \qquad (2.01)$$

This means that there is *certainly* a case in which E holds, and that is when x is y: whereas it is *possible* that E may hold when x is not y, or y not x; or when neither x nor y co-exist with it.

We have thus in the case of continuity or self-contradiction to fall back upon the equations already employed in the case where $x.(1-x) \neq 0$ and $x = x^n$, where n is greater than 2. In the long series extending to an indefinite degree the identity of x with x may become altered, and such is the Hegelian assumption or contention which all Aristotelians regard as fallacious, but which is the very essence of the Dialectic process. We see that like the Miksowski theory of space it is in fact a necessary concomitant of all algebraic logic. Without it there is a breach in the whole process of thought. However self-sufficient the analysis on the assumption of a perfect identity, thought requires an ultimate continuity between an object and its negation, a continuity that could be attained only by a change in the identity series of the equation $x = x.x.x \ldots x.x. = x^n$, whether in time or space, or in mere logical abstraction; not merely in the 'forms of intuition' upon which our conceptions of space and time depend, but in the other categories of the understanding, which need not be limited to twelve, as Kant supposed,

but as Hegel has shown may admit of an indefinite number of ideas of the nature of categories.

It would no doubt be said that by no stretch of imagination could it be conceived that the mere repetition of an idea could alter it from being ever and always the same, identically the same as in De Moivre's Theorem. The psychologist would hardly agree with that view. But be that as it may, the algebraic analysis not only suggests it but requires it. The beauty of the algebraic analysis is that the identity is preserved whilst the transition from the object to negation becomes possible. Difficult as the algebraic facts are to translate into familiar terms they are not more difficult than the facts of ordinary experience.

The interpretation of the logical equations by Boole's method, quite apart from their application, which as we have seen can as a rule be replaced by others of a far simpler nature, yet affords material of a most fascinating nature particularly as there is no definite explanation forthcoming for some of the algebraic processes that give so admirably the correct logical results without in any way satisfying the understanding, as ordinary logic might do. For instance what can $\frac{0}{0}$ or $\frac{1}{0}$ really mean in all these operations if notwithstanding their unintelligibility they carry us through the most intricate calculations to the right conclusions? Like the mysterious $\sqrt{-1}$, which is purely imaginary, they lead us to the correct results. Can it be that we are here on the threshold of the Kantian interpretation of the categories? And that in the alogarithmic we have the more general laws of thought, which in virtue of the particular forms of intuition of the understanding thereby limit the operations of the reason? The true aspect of things remains hidden from our minds by the mere channels of perception. Something, it may be said, there must be in all this that half conceals and half reveals both the meaning and the soul of things!

Let us consider for example the proposition 'All x is y'. This may be written $x = v.y$; and eliminating v, by putting it successively equal to 1 and 0 respectively, and then combining the two results, we obtain $(x-y).x = 0$; or since $x^2 = x$, we get $x.(1-y) = 0$, i.e. $x.\bar{y} = 0$; consequently by the ordinary rules of algebra, the precise meaning of which is, as we say, by no means clear in the present case, except that experience proves it to be justified,

$$x = \frac{0}{\bar{y}} = \frac{0}{0}.y + \frac{0}{1}.\bar{y}. \qquad (2.02)$$

Again, if 'No x is y' or $x.y = 0$, then $x = \frac{0}{y} = \frac{0}{1}.y + \frac{0}{0}.\bar{y}$. (2.03)

Again, if

$$x.y = z, \text{ then } x = \frac{z}{y} = \frac{1}{1}.y.z + \frac{0}{1}.y.\bar{z} + \frac{1}{0}.\bar{y}.z + \frac{0}{0}.\bar{y}.\bar{z}. \tag{2.04}$$

But if $x.y.\bar{z} = 0$, then there is an extra term $\frac{0}{0}.\bar{y}.z$ to be introduced. These results are as beautiful as they are curious, and well worthy of investigation.

The rigour of the reasoning becomes particularly apparent in the following example. Let x = one who believes in the Popes, y = a Christian, z = a Catholic. Then since 'Every Catholic believes in the Popes and is a Christian, whilst conversely every Christian who believes in the Popes is a Catholic', it follows that $xy = z$.

Consequently

$$x = \frac{z}{y} = \frac{1}{1}.yz + \frac{0}{1}.y\bar{z} + \frac{1}{0}.\bar{y}z + \frac{0}{0}.\bar{y}\bar{z}. \tag{2.05}$$

Hence those who believe in the Popes are either both Christians and Catholics or else neither one nor the other. This astounding result, though strictly logical, shows that the definition which at first sight seemed adequate was really at fault. For we know that to believe in the Popes and yet be neither a Catholic nor a Christian is incongruous. The explanation is obvious, for the definition states that only those Christians who believe in the Popes are Catholics; but it states nothing whatever about those Christians who do not believe in the Popes, except that they are not Catholics; and for the same reason it ignores though it does not deny the possibility of believers in the Popes that are not Christians; since the definition of Catholics given tells us nothing about them. This shows how easily one can be caught napping in making definitions. To the definition must be added, what it does not as such imply, belief in the Popes requires belief in Christianity or else Catholicism $x\bar{y} = 0$ or $x\bar{z} = 0$; for if they had been the Vicars of Mohammed the inference would never have been made other than as the algebraic solution requires.

Hence $x\bar{y} = 0$. Therefore we have $(xy-z)^2+(x\bar{y})^2 = 0$, or

$$x(1-2yz)+z = 0,$$

i.e. $$x = \frac{z}{(2yz-1)} = \frac{1}{1}.yz + \frac{0}{-1}.y\bar{z} + \frac{1}{-1}.\bar{y}z + \frac{0}{-1}.\bar{y}\bar{z}; \tag{2.06}$$

so that now since all who believe in the Popes believe in Christ they are all Catholics.

Likewise $$y = \frac{(x+z)}{2xz} = \frac{2}{2}.xz + \frac{1}{0}.x\bar{z} + \frac{1}{0}.\bar{x}z + \frac{0}{0}.\bar{x}\bar{z}. \qquad (2.07)$$

A Christian therefore might not, so far as the definition goes with the extra information given, believe in the Popes, but he would obviously not in that case be a Catholic.

The correct definition of course is $z = x$; and $z\bar{y} = 0$.

Therefore $$xyz + \bar{x}\bar{z}(y+\bar{y}) = 1. \qquad (2.08)$$

I give this example for the benefit of my Aristotelian friends who may accuse me of shooting a fly with a cannon. I have no doubt that many a fly has been shot by a cannon, though heavy artillery is really intended for more formidable objects, and is often very effective as such.

CHAPTER V

THE CONJUGATE UNIVERSES OR ASPECTS OF TRUTH

§ 1

THE resemblance between this logical indeterminateness and some of the anomalies of a more difficult nature, as in non-Euclidean geometry, is also worth noting. But the actual relationship in this respect will be dealt with in its proper place. Sufficient is it here to point out that $\frac{0}{0}$ and $\frac{1}{0}$ as logical coefficients occupy a striking position in this mathematical logic.

In general a class or proposition w, that can be expanded as a function of a number of other classes or propositions x, y, z, &c., may be written*

$$w = \frac{1}{1}.A + \frac{0}{1}.B + \frac{0}{0}.C + \frac{1}{0}.D. \qquad (2.09)$$

This breaks up into the separate equations

$$w = \frac{1}{1}.A + \frac{0}{0}C; \text{ and } D = 0; \qquad (2.10)$$

since $\frac{0}{1} = 0$; and $\frac{1}{0}$, though supposed to be infinite, does not enter into the logical equation of the Booleian calculus. The term for which it stands as coefficient is supposed to vanish.

If 'every x that is y' is also z, then $x.y.\bar{z} = 0$; or

$$x = \frac{0}{y\bar{z}} = \frac{0}{0}.yz + \frac{0}{1}.y\bar{z} + \frac{0}{0}.\bar{y}z + \frac{0}{0}.\bar{y}\bar{z}; \qquad (2.11)$$

whereas if it be stated that at the same time 'every z is also $x.y$', so that xy and z become identical, and the two propositions may be expressed by the single equation $xy = z$, then

$$x = \frac{z}{y} = \frac{1}{1}.yz + \frac{0}{1}.y\bar{z} + \frac{1}{0}\bar{y}z + \frac{0}{0}.\bar{y}\bar{z}; \qquad (2.12)$$

hence we have

$$x = \frac{1}{1}.yz + \frac{0}{0}.\bar{y}\bar{z}; \text{ and also } \bar{y}z = 0;$$

i.e. that 'every z is y' is implied in the original proposition, which

* See Boole, Chapter VII, loc. cit.

in general requires that 'every x is either both y and z or else possibly neither one nor the other'.

Again if 'every xz is yz, and vice versa, it does not follow algebraically any more than logically that x is y, although at first sight one might be inclined to draw this conclusion from the equation $z.x = z.y$; that $x = y$ for correctly $z.(x-y) = 0$; or

$$x = \frac{zy}{z} = \frac{1}{1}.yz+\frac{0}{0}.y\bar{z}+\frac{0}{1}.\bar{y}z+\frac{0}{0}.\bar{y}\bar{z}; \qquad (2.13)$$

so that $x = yz+\bar{z}.(y+\bar{y})$, that is, x is either both y and z or else not z at all, although it may or may not be y in such a case. Otherwise if it is y, it may or may not be z; but if it is not y, then it cannot be z. All this though obvious in the ordinary algebra and logic is not so clearly emphasized thereby.

In the Booleian equation that we have been considering the meaning to be attached to the various types of coefficients is as follows:

(1) $\frac{0}{1}$, Non-existent; (2) $\frac{1}{0}$, Impossible; (3) $\frac{0}{0}$, Possible; and (4) $\frac{1}{1}$, Actual. (2.14)

Thus they correspond to the Kantian categories of modality.

The interpretation of the coefficients in the Booleian equations may be summarized thus:

If the solution of a problem, obtained by development, be of the form

$$w = A+0.B+\frac{0}{0}.C+\frac{1}{0}.D, \qquad (2.15)$$

the solution thus obtained may and must be resolved into the two following equations:

$$\begin{aligned} w &= A+v.C \quad . \quad . \quad . \quad (1) \\ D &= 0 \quad . \quad . \quad . \quad . \quad (2) \end{aligned} \qquad (2.16)$$

v being an indefinite class symbol.

D is in fact the negation of w. Its absence renders the proposition relating to w as expressed by A and C possible. The latter may be described as finite truth; the former as the truth at infinity as its coefficient is infinite, and does not enter into finite affairs. It is the contradiction of the finite truth which must be regarded as of the whole in the totality of things. Just as in geometry it is permissible to speak of the point, or the line, or the plane at infinity, so algebraically would it be permissible to refer to the truth at infinity; meaning thereby the contradictory of the truth as we have

it. The two make up the totality. At first sight there appears a curious paradox in regard to the method of solution employed in p. 84. If $V_1 = 0$, $V_2 = 0$, &c., be the functions of x, y, z, &c., as premisses, they obey the law of contradiction $V_1(1-V_1) = 0$, $V_2(1-V_2) = 0$. If $V^2 = V_1^2+V_2^2+\ldots$, then in the expansion of V in terms of x, y, z, &c., we have $V = 0.P+1.Q$; the complete solution of the problem being given by the equation $P = 1$, $Q = 0$ representing the premisses. This is quite obvious from what we have just seen in the case of the solution $w = A+v.C$, and $D = 0$; for $V = D$ represents the negation of the premisses; so that that which is negated in V is the truth, namely the terms whose coefficients in V are $= 0$. We thus perceive why it is that in order to obtain the complete solution of the problem $V = 0$, where V is the sum of all the separate propositions constituting the premisses, each negated and equated to zero, we have merely to pick out all the terms whose coefficients are zero in the expansion of V and equate them to unity. The negation of the truth when negated is the truth.

All this of course assumes that the law of contradiction holds for each of the propositions V_1, V_2, &c.

It must be borne in mind that in the equation

$$w = \frac{1}{1}\cdot A+\frac{0}{1}\cdot B+\frac{0}{0}\cdot C+\frac{1}{0}\cdot D,$$

although $D = 0$ is assumed, yet it does not follow that $B = 0$; nor is it assumed that such is the case. But $w.B = 0$; so that $B = 0$ only when it is combined with w; whereas we have $D = 0$ independently of w. This is a most important distinction. $D = 0$ is the negation of the universal, and $w.B = 0$ that of the particular, i.e. of the major and minor premisses respectively.

Thus if $x\bar{y} = 0$ and $y\bar{z} = 0$, then $x\bar{y}+y\bar{z} = 0$, and therefore

$$x = -\frac{y\bar{z}}{\bar{y}} = \frac{0}{0}\cdot yz+\frac{1}{0}\cdot y\bar{z}+\frac{0}{1}\cdot \bar{y}z+\frac{0}{1}\cdot \bar{y}\bar{z}, \qquad (2.17)$$

or $x = \frac{0}{0}\cdot yz$; and as given $y\bar{z} = 0$. But it is not true that either $z\bar{y} = 0$, or $\bar{y}\bar{z} = 0$, although these terms do not enter into the solution. It is only when they are associated with x that they vanish, i.e. only $x\bar{y}z = 0$ and $x\bar{y}\bar{z} = 0$, for $x\bar{y} = 0$ independently of z.

Again, if $xy = z$, then

$$x = \frac{z}{y} = \frac{1}{1}\cdot yz+\frac{0}{1}\cdot y\bar{z}+\frac{1}{0}\cdot \bar{y}z+\frac{0}{0}\cdot \bar{y}\bar{z}. \qquad (2.18)$$

Here $x\bar{y} = 0$ is obviously true, whereas $y\bar{z}$ is not $= 0$, unless it be with reference to x, for only $xy\bar{z} = 0$.

In dealing with the equations $D = 0$ and $w.B = 0$ we are naturally reminded of the analogous case in geometry. Thus any two ellipses will intersect in four points, real or imaginary. Now circles are particular cases of ellipses whose major and minor axes are equal. But no circles can intersect in more than two real points; and they must cut each other in two real points if they cut each other at all, except when they merely touch, and the two points coincide. In all such cases, however, they are regarded as cutting each other in two other points at infinity, called I and J.

In the algebra of logic as here treated the equation $D = 0$ is the negation of the truth at infinity or the opposite world of negation. In other words, the truth at infinity is a lie here. And conversely, a truth here is a lie at infinity. As in non-Euclidean geometry, what is true here need not be true everywhere and always. The law of contradiction is akin to the Euclidean definition of parallels. Contradictories can merge into one at infinity, and conversely, since contradictories do merge here, the Absolute Truth in regard to them is at infinity.*

Thus if we adopt Dr. J. H. Venn's graphic method of representing classes by circles or ellipses or other closed curves we must regard all classes which exclude each other here as intersecting at infinity, so that the laws of non-Euclidean space could be applied to his logic without appearing in the least absurd. 'The rich are not poor' becomes 'some rich are poor' and 'some poor are rich', at infinity. It cannot be said, however, that even at infinity 'all rich are poor', or 'all poor are rich'. The conception of infinity is what gives self-sufficiency to thought. If the thing is to be taken seriously, it means of course that every truth has its contradictory at infinity; actually and not merely fictitiously so. For Art is the antithesis of imperfection.

The resemblance, however, between this logical system and non-Euclidean geometry does not justify our pushing the analogy too far. It is true that as Whitehead and Russell have developed the subject, geometry becomes the basis of logic. And it is difficult to see where the difference between them, if any, is to be found. If all entities are points, and classes lines, all logical discussions might be transferred to questions of the properties of points and lines;

* The arithmetic of infinite numbers worked out by Cantor and Frege is dealt with elsewhere (*vide* p. 209). It solves the problem of infinity and continuity and, as Russell remarks, may be regarded as perhaps the greatest achievement of our times. It must be admitted to be of a different nature from what we are now considering. That a part is equal to the whole is true only of this infinity.

in other words geometry. Dr. Johnson is said to have complained in a moment of exasperation in a controversy, that the discussion had degenerated into mere questions of grammar and syntax.* That it should be resolved into a question of formal logic seems hard enough; but that it should wind up with problems in the geometry of hyperspace may seem intolerable, unless perhaps steps are taken to make things worse by the use of the higher algebra! Yet what is to be done if such be really the issue and secure that the last error shall not be worse than the first? We have seen that it would be quite impossible by the unaided intellect to arrive at some of the results we have obtained, with such striking simplicity, or indeed to master the general principles involved.†

The derivation of the properties of space of any number of dimensions whatsoever from general ideas relating to classes, and likewise the laws of thought from similar principles, should nevertheless keep us on our guard against identifying the properties of thought with those of space.

The discontinuity between reality and unreality, truth and falsehood, right and wrong, conscious and unconscious, brings us back to the problems already discussed, that the law of contradiction as distinct from that of identity is confined to extremes, whereas the latter from that which is to that which is not, like the development of the simple chrysalis into the gorgeous golden-coloured and glittering butterfly. What splendour has not evolved from the simplest and most ordinary beginnings, whether in the immediate or the remote past? What animal of the most beauteous type was not once a mere germ and an ovum? Can dust and ashes ever reveal the mystery of being with greater force and eloquence than the microscope and test-tube have disclosed to us? Yet if the Higher Self is but an idea as it must be admitted to be, it has evolved in the race, as in the individual, from spacial microcosmic particles of life-stuff, ultimately resolvable, into elements of those forms of sensuous intuition we call space and time by which alone our experience of the world as something apparently independent of ourselves becomes possible. It is thus also obvious that such contraries as life and death stand for opposites, that these are true or untrue, real

* He might have added that grammar is the last refuge of a cynic.

† Needless to say it is very much easier than to acquire a thorough knowledge of Greek grammar, and it can also be learnt in less time than a modern language such as German. It may even rank in its simplicity with Italian or Spanish. Although memory does not play so important a part, owing to the absence of a vocabulary, yet imagination and understanding, as well as ratiocination, are drawn upon to a greater extent than the study of grammar and syntax actually demands.

or unreal, according to the world they refer to in the universe of discourse. That which is living here and now is not living elsewhere and at another time. And that which is not living here and now may and according to this theory *must* be living elsewhere and for ever both before and after, if ideas have any meaning and the logic of algebra serves to complete with perfect harmony the meaning of the whole.

Let it not be forgotten for a moment that when we speak of here and there we imply that the terms so spoken of are in relation to space; but not here or not there may exclude space altogether, i.e. $\bar{s}$, a perfectly legitimate assumption from a purely logical point of view; since as we have said, the factor $(s+\bar{s})$ enters into the equation of the universe of discourse; as indeed the similar factor of $(t+\bar{t})$ in regard to time must be taken into account. Hence the absence of space and time must not logically be disregarded. The question is whether the mere logical possibility necessarily entails its actuality; for the negation would imply the truth negated in the negated world of otherness.

The coefficient $\frac{1}{0}$ which precedes the term D in the universe of discourse implies that $D = 0$, if $w = A+v.C$. This would seem to follow from the general equation

$$w = \frac{1}{1}.A+\frac{0}{1}.B+\frac{0}{0}.C+\frac{1}{0}.D.$$

Supposing that we grant that $D = 0$ for s, i.e. $Ds = 0$, yet it does not follow that $D\bar{s} = 0$: on the contrary $D\bar{s}$ is not 0 unless $D = 0$. Now as we think in space we have to deal with Ds and not with D independently of s. This is Kant's theory expressed in mathematical form.

We write $F(xy) = x.y+x.\bar{y}+\bar{x}.y+\bar{x}.\bar{y} = 1.$

This may in general be expressed in the form of the most general expansion

$$F(x,y) = \frac{1}{1}.A+\frac{0}{1}.B+\frac{0}{0}.C+\frac{1}{0}.D. \qquad (2.19)$$

Now if 'no x is y', i.e. $x.y = 0, x.\bar{y}+\bar{x}.y+\bar{x}.\bar{y} = 1,$

or $$F(x.y) = \frac{1}{1}.A+\frac{0}{0}.C; \text{ and } D = 0. \qquad (2.20)$$

D in fact is $x.y$. And in general if D is $f(x, y, z,\ldots)$ then the terms that vanish in the expansion of this latter are in fact the real terms in $F(x.y)$; that is, if $D = \frac{1}{1}.A'+\frac{0}{1}.B'+\frac{0}{0}.C'+\frac{1}{0}.D'$ we must (2.21)

have $B'+D'$ equivalent to $A+C$ in $F(xy)$; and $A'+C'$ equivalent to $B+C$; that is, the real terms in one are the imaginary in the other, and the contradictory of the real. D and D' are what might be called conjugate terms, contradictory of each other, so that $D+D' = 1$.

Somewhere, somehow, the contradictory or negation of all that is is, i.e. $D+D' = 1$. This apparently nonsensical, or somewhat humorous, proposition gives unity and harmony to the whole, without which all is one-sided and unmeaning, lacking in ballast, completion, and logical unity in the highest sense of totality. It does not mean that the contradictory of any proposition here and now must be, need be, or can be accepted here and now. But on the contrary must it be positively denied for here and now. Only somehow, and not here and now, and not necessarily anywhere nor at any time, but in some transcendental sense, beyond space and time, beyond cause and effect, beyond good and evil, this algebraic solution of the problem is in accordance with the Kantian transcendental of pure reason; and also with the Hegelian notion, likewise transcendental. In the transcendence of all contradiction lies the reality. The laws of the understanding apply only to D or to D', at a time; but never to both at once. What Hegel calls the 'Reflexion' is here exemplified as the contradictory contradicted. Further the 'Ground' or higher synthesis is obtained by this process of negation, as by a spiral staircase towards the lofty regions of the Absolute.

§ 2

Though $f(x)$ may be expanded in an infinite series of ascending powers of x, by Taylor's or Maclaurin's theorems, yet it may have a finite value in itself. The function is not necessarily infinite. This point should never be lost sight of. The analogy between number and mere class regardless of number is thus apparent. But what we have proved holds for one as for the other.

$$\text{In general } f(x,y,z,w,\ldots) = \frac{1}{1}\cdot A+\frac{0}{1}\cdot B+\frac{0}{0}\cdot C+\frac{1}{0}\cdot D. \quad (2.22)$$

Therefore, $f(x, y, z, w, \ldots) = A+\frac{0}{0}\cdot C$, and $D = 0$; D should be included if the law of contradiction does not hold as in dialectic, but not otherwise, as in strict logical processes with an excluded middle.

If

$$D = \frac{1}{1}.A' + \frac{0}{1}.B' + \frac{0}{0}.C' + \frac{1}{0}.D' = 0; \text{ then } B' = A + v.C. \tag{2.23}$$

Similarly $B = A' + v.C'$; D and D' may be said to be conjugate functions. One may be considered as passing alternately into the other.

An interesting case is the following in which, by supposing a common element between opposites, we are led by negation back to where we started. Thus, if x is y, i.e. $x = yz$, then

$$z = \frac{x}{y} = \frac{1}{1}.xy + \frac{1}{0}.x\bar{y} + \frac{0}{1}.\bar{x}y + \frac{0}{0}.\bar{x}\bar{y} = xy + \bar{x}\bar{y}; \tag{2.24}$$

that is, the class z may be both x and y or neither one nor the other. This is, as we have seen, obvious, since it cannot be one or the other separately; for if it be x it must be y, and every z that is y is also x. But z may not be y at all, in which case it would be neither x nor y, there being only four possible cases. We may regard (xy) as the 'quotient' and $\bar{x}\bar{y}$ as the 'remainder' in the process of division of x by y.

Now following the procedure of obtaining the greatest common measure or divisor in arithmetic, we suppose that the two opposites xy and $\bar{x}\bar{y}$ have a common factor, and repeat the process of division; thus, we obtain:

$$w = \frac{xy}{\bar{x}\bar{y}} = \frac{1}{0}.xy + \frac{0}{0}.x\bar{y} + \frac{0}{0}.\bar{x}y + \frac{0}{1}.\bar{x}\bar{y} = x\bar{y} + \bar{x}y. \tag{2.25}$$

Namely the quotient and remainder of xy and $\bar{x}\bar{y}$ must be either x or y separately since these are the only possible alternatives to being both or neither. One may be regarded as the quotient and the other the remainder.

But if we repeat the process we find that we shall return to where we started from.

$$c = \frac{x\bar{y}}{\bar{x}y} = \frac{0}{0}.xy + \frac{1}{0}.x\bar{y} + \frac{0}{1}.\bar{x}y + \frac{0}{0}.\bar{x}\bar{y} = xy + \bar{x}\bar{y}. \tag{2.26}$$

This is equivalent to the Hegelian dialectic in which c may be regarded as the concept arrived at by the synthesis. c is not precisely the same as z; for in the latter the coefficient of xy is $\frac{1}{1}$, and in the former it is $\frac{0}{0}$. If

$$xy = \frac{1}{1}.A + \frac{0}{1}.B + \frac{0}{0}.C + \frac{1}{0}.D, \tag{2.27}$$

then in z, xy is A; but in c, it is C. Thus in fact

$$c = v.(xy+\bar{x}\bar{y}). \tag{2.28}$$

This result is well worth noting. The particular is now confined not merely to $\bar{x}\bar{y}$, as in z, but also to xy itself, which is thus more limited in its nature.

Hence by the process of negation, and negation of the negation, we refine the class xy, or improve z when transforming it into c.

If we write $p = xy$ and $q = \bar{x}\bar{y}$, then if $l.q = p$, we have

$$l = \frac{p}{q} = \frac{1}{1}pq+\frac{0}{0}\bar{p}\bar{q} = xy.\bar{x}\bar{y}+\frac{0}{0}(1-xy)(1-\bar{x}\bar{y})$$

$$= \frac{0}{0}(1-xy-\bar{x}\bar{y}) = \frac{0}{0}x\bar{y}+\frac{0}{0}\bar{x}y,$$

since $xy+x\bar{y}+\bar{x}y+\bar{x}\bar{y} = 1$ as before.

Now if we put $r = x\bar{y}$ and $s = \bar{x}y$, and suppose that $ms = r$, we arrive at the result

$$m = \frac{1}{1}rs+\frac{0}{0}\bar{r}\bar{s} = x\bar{y}\bar{x}y+(1-x\bar{y})(1-\bar{x}y)$$

$$= \frac{0}{0}(1-x\bar{y}-\bar{x}y) = \frac{0}{0}(xy+\bar{x}\bar{y}). \tag{2.29}$$

Therefore $m = \frac{0}{0}.z$. m in fact is what we previously called c.

Since $c = v(xy+\bar{x}\bar{y})$ it follows that $c.(1-xy-\bar{x}\bar{y}) = 0$, in other words $c.(x\bar{y}+\bar{x}y) = 0$. Hence $x(c\bar{y}-cy)+cy = 0$; or

$$x = \frac{cy}{cy-c\bar{y}} = \frac{1}{1}.cy+\frac{0}{-1}.c\bar{y}+\frac{0}{0}.\bar{c}y+\frac{0}{0}..\bar{c}\bar{y} \tag{2.30}$$

Otherwise, since $x = yz$, and $y+\bar{y} = 1$,

$$x = \frac{1}{1}.cy+\frac{0}{0}.\bar{c} = zy.$$

$$\therefore\ z = \frac{cy+\frac{0}{0}\bar{c}}{y} = \frac{1}{1}cy+\frac{0}{0}c\bar{y}+\frac{0}{0}\bar{c}y+\frac{0}{0}\bar{c}\bar{y}. \tag{2.31}$$

In other words $z = \frac{1}{1}cy+\frac{0}{0}c\bar{y}+\frac{0}{0}\bar{c}y$ if $\frac{\frac{0}{0}}{0}$ is infinite,

and as $x = zy$, $x = zy = \frac{1}{1}cy+\frac{0}{0}\bar{c}y = \left(\frac{1}{1}c+\frac{0}{0}\bar{c}\right)y.$

$$\therefore\quad c = vz.$$

This is of importance in Peano's distinction between $x = \epsilon y$ and $x \supset y$, evidently $\frac{1}{1}c$ is taken distributively; $\frac{0}{0}c$ collectively.

§ 3

We are now in a better position to consider the manner of *emergence* and development in the evolution of ideas, whether of Nature or of thought. In general if $x_1, x_2, x_3, \ldots x_n$ be the classes that can form the subjects or objects of a proposition; and if n be infinite, so as to include the whole universe of ideas of every possible kind howsoever contradictory they might be in some cases with each other, then such a heterogeneous congeries forms what might in other words be regarded as chaos from which all begins. We may therefore write for the relationship between the various elements,

$$F(x_1, x_2, x_3, \ldots x_n) = 1,$$

or
$$F = (x_1 + \bar{x}_1)(x_2 + \bar{x}_2)(x_3 + \bar{x}_3)\ldots(x_n + \bar{x}_n) = 1, \qquad (2.32)$$

where one of the x's corresponds to the element of time, which we may otherwise express by t. This t will consist of $t_1, t_2, t_3, \ldots t_n$, where n has all values between $+$ and $-$ infinity.

Now with the progression of time terms keep dropping out such as, let us say,

$$x_m\bar{x}_n = 0,\ x_r\bar{x}_s = 0,\ \ldots, \quad \text{and} \quad x_u x_y = 0,\ x_y x_z = 0,\ \ldots.$$

Represented in this manner the history of each monad would present the appearance of the familiar pianola-music. From what we have seen F does not represent the Totality, but only the terms $A+C$. It stands only for $\bar{D}$, as the negation of $D = 0$, so that $\bar{D} + \bar{\bar{D}} = 1$. Hence when the various propositions $x_r\bar{x}_s = 0$, $x_u x_v = 0$, vanish from one side they are transferred to the conjugate function. The two worlds then are the reverse of each other, like the photographic negative and its positive print. From the very nature of the idea what is present or true in one universe is necessarily absent in the other. What is living here is dead there. And what is dead here is living there. But that which is illusion here may be illusion there in its contradictory, if it is not real. Only what is definitely true and capable of definite negation is negated, or can be so. Hence appearance is excluded from D and $\bar{D}$ separately, and must be regarded as belonging to a group apart, $D.\bar{D}$, which belongs to both.

Algebraically it is so expressed. But the algebraic continuity in geometry is that which renders analytical geometry possible, and

it should do so likewise in Logic. Appearance is thus the connecting link between the real and the unreal; as it is the bond between truth and error. The relationship may be represented thus:

$$D+D.\bar{D}+\bar{D}=1.* \qquad (2.33)$$

And if $D.\bar{D}$ consists of a gradual change from D to $\bar{D}$, we may write $D+D_1\bar{D}_1+D_2\bar{D}_2+...+\bar{D}=1.$

If the judgements of various persons as to the relative truths be $\mathcal{J}_1, \mathcal{J}_2, ...$, then we may regard

$$\mathcal{J}_1 = D_1\bar{D}_1, \quad \mathcal{J}_2 = D_2\bar{D}_2, ..., \quad D+\mathcal{J}_1+\mathcal{J}_2+...+\mathcal{J}_n+\bar{D}=1. \qquad (2.34)$$

Ordinary judgements, however they differ, are in general in the middle terms between the two extremes at infinity, that apparently are united in one. D and $\bar{D}$ are ultimately united. And although as we have seen D and $\bar{D}$ are mutually exclusive in reality, yet in every concrete case of y that is not x, the negation is contradicted at infinity, in some transcendental sense.

In the conduct of life we must take some definite course of action, and assume some particular practical belief as a rational basis or guidance for our behaviour and pursuit. All may have the same object, but adopt different paths; some indeed diametrically opposite ones; like the rival travellers from Dublin to London, one of whom chose to go via Holyhead, and the other decided to get younger by taking the course via New York and Yokohama, insisting, no doubt rightly from his own point of view, that he had been a day less on his journey than his rival believed him to have been. Notwithstanding the length of his absence, he had gained a day by his antics. The wrong course has some element of truth and some shadow of compensation. The penalty for a step in the wrong direction may not be as fatal as it may seem to be. But in every case if the final aim be Truth it may yet be reached the more quickly by those whom we consider to have turned their backs to it; and in any case have collected treasures of untold value on their prolonged and apparently irrelevant path. They will at least have enlarged their experience and realized the immensity of the

* If D and $\bar{D}$ be regarded as the Aristotelian or Thomist antitheses of Mind and Matter, the psycho-parallelism between the two becomes manifest. Matter may be regarded as the negation and concomitant of mind, the equivalent of the monad; or otherwise mind is negative matter. Karl Pearson and Sir Arthur Schuster have pointed out the probability of a Universe of negative matter, that is according to the above, mind or monad. Dr. Whitehead's conception of God and the World in the last chapter of *Process and Reality*, which appeared since the above was written, would seem to be somewhat of this nature, though arrived at from different considerations.

world of their adventures, whilst their more common-sense rival in his directness retains his insular notions of the great world around him in which he plays a narrow useful but perhaps insignificant part.

We may, before going more fully into the Hegelian dialectics, examine some of the simple systems that preceded his, based upon the law of contradiction and excluded middle.

CHAPTER VI

PRE-HEGELIAN IDEALISM

§ 1

SCHELLING endeavoured to arrive at results which though refined seem very much open to question; concluding that the real is at once both rational and true; and that the unreal is irrational and untrue. We may proceed upon similar lines as follows: If R = the real; R_1 = the rational; and T = the true.* The assertion amounts to

$$R.R_1.T+\bar{R}.\bar{R}_1.\bar{T} = 1. \qquad (2.35)$$

In order that this should be the case we must obviously have that

$$T.(R.\bar{R}_1+\bar{R}.R_1+\bar{R}.\bar{R}_1) = 0, \qquad (2.36)$$

and

$$\bar{T}.(R.R_1+R.\bar{R}_1+\bar{R}.R_1) = 0.$$

In general four propositions are necessary to establish the general proposition consisting of but two terms as above; the other being necessary consequences of the premised four.

It is possible, however, to arrive at the required result from merely three propositions, for if we admit that (*a*) the true is rational, i.e. $T.\bar{R}_1 = 0$; and that (*b*) the real is true, i.e. $R.\bar{T} = 0$, we obtain that

$$R.T.R_1+\bar{R}.\{T.R_1+\bar{T}.(R_1+\bar{R}_1)\} = 1. \qquad (2.37)$$

Then if, as Schelling assumes, 'the rational is real', i.e. (*c*) $R_1.\bar{R} = 0$, it follows that

$$R.T.R_1+\bar{R}.\bar{T}.\bar{R}_1 = 1.$$

If, on the other hand, we had merely asserted that 'the true is real', i.e. $T.\bar{R} = 0$, then we should have had

$$R.T.R_1+\bar{R}.\bar{T}.(R_1+\bar{R}_1) = 1, \qquad (2.38)$$

so that the rational would not necessarily be either true or real.

Hence from the above premisses (*a*), (*b*), and (*c*) we can conclude that 'the real is rational', 'the true is real', and 'the rational is true'. None of these three could have been taken as fundamental assumptions, though Schelling's School apparently regarded the first of these as such; and *its converse*, that the rational is real and the

* The same symbols must always be understood to be of the same type, else the reasoning is fallacious as already explained. The various symbols, however, may all be of different types.

real rational, i.e. $R.\bar{R}_1 = 0$, and $\bar{R}.R_1 = 0$. But he leaves the possibilities that $T.\bar{R}.\bar{R}_1$ and $\bar{T}.R.R_1$ might both co-exist, i.e. 'the true might be neither rational nor real' and 'the rational and real might not be true'! It would be necessary that the two outstanding terms should vanish by two of six combinations. It might be supposed that the argument is nothing more than a *petitio principi*; for the assumption that the rational is real, i.e. $R_1.\bar{R} = 0$, appears to be of this nature.

That the rational is real, however, follows from the fact that the rational must be realized; and that that which is realized is real. Thus if $r =$ that which is realized, then $R_1.\bar{r} = 0$, $r.\bar{R} = 0$; $\therefore R_1.\bar{R} = 0$. Strictly speaking, $R_1.r.\bar{R} = 0$.

In virtue of the fact that whatever is realized is real, i.e. $r.\bar{R} = 0$, we have $[rR(R_1+\bar{R}_1)+\bar{r}\bar{R}_1(R+\bar{R}_1)] = 1$, bearing in mind of course that the rational must be realized, i.e. $R_1\bar{r} = 0$. Consequently it inevitably follows, provided $T\bar{R}_1 = 0$ and $R\bar{T} = 0$, that

$$R.T.R_1.r+\bar{R}.\bar{T}.\bar{R}_1.\bar{r} = 1. \qquad (2.39)$$

Therefore that which is realized is true, rational, and real; and that which is not so realized is none of these.

The premisses are:

(1) That which is true is rational, i.e. $T.\bar{R}_1 = 0$.

(2) Whatever is rational is such only when it is realized, i.e. $R_1.\bar{r} = 0$.

(3) That which is realized is real, i.e. $r.\bar{R} = 0$. (2.40)

(4) That which is real is true, i.e. $R.\bar{T} = 0$.

Therefore $T = R_1 = r = R = T$, and $\bar{T} = \bar{R}_1 = \bar{r} = \bar{R} = \bar{T}$.

Hence what is not realized is neither rational, real, nor true.

If, however, $R_1.\bar{r} \neq 0$, that is, that the rational may not be realized, it follows that the rational may or may not be real; and may or may not be true. Likewise the true may not be real; although the real is both rational and true, even when it is not actually realized. The proposition $R_1.\bar{r} = 0$, i.e. the rational is something realized, appears to be the *pièce de résistance* in such a system of philosophy.

If we put $X_0 = \bar{R}.\bar{T}.R_1.r$; and $\bar{X}_0 = \bar{R}.\bar{T}.\bar{R}_1.\bar{r}$, then $X_0+\bar{X}_0 = 1$ represents Schelling's dilemma. In so far as an illusion, delusion, or other form of error could not be true, they could not be adequately realized. A person under a delusion cannot be said to realize the truth adequately; and in order to avoid the obvious difficulty in this respect the term 'realized' must be defined so as to exclude ambiguity.

If we put, as on previous occasions, $c =$ certainty and $d =$ a

delusion, then since $c.\bar{r} = 0$, i.e. that which is certain is realized, $T.d = 0$, Truth is not a delusion.

Then, combining these two propositions with the dilemma

$$R.T.R_1r+\bar{R}.\bar{T}.\bar{R}_1\bar{r} = 1, \text{ or } X_0+\bar{X} = 1,$$

we readily obtain that

$$X_0.\bar{d}.(c+\bar{c})+\bar{X}_0.\bar{c}.(d+\bar{d}) = 1; \tag{2.41}$$

since $$X_0d = 0, \text{ and } \bar{X}_0c = 0.$$

It follows that (*a*) $c.d = 0$; (*b*) $X_0.d = 0$; (*c*) $c.\bar{X}_0 = 0$; (*d*) $X_0.d.c = 0$; (*e*) $r.d = 0$; (*f*) $R_1.d = 0$; (*g*) $r\bar{R}_1 = 0$.

Hence it is quite clear that the word 'realized' must be used so as to exclude all that is irrational; for a delusion cannot be realized, in this sense, and what is rational is not a delusion. In fact, writing the general equation (2.41) in full we have

$$R.T.R_1r\bar{d}.(c+\bar{c})+\bar{R}.\bar{T}.\bar{R}_1.\bar{r}.(d+\bar{d}).\bar{c} = 1, \tag{2.42}$$

from which it appears that whatever is realized is rational and true, although it may not be certain even as such.

It is still thus possible so far as the argument goes that an idea which is no delusion may nevertheless not be true or rational. And if this be denied, as it obviously should be, then must $\bar{T}.\bar{d} = 0$, i.e. if an idea is not true it is a delusion; and vice versa, if it is not a delusion it must needs be true. In other words

$$R.T.R_1.r.\bar{d}.(c+\bar{c})+\bar{R}.\bar{T}.\bar{R}_1.\bar{r}.\bar{d}.\bar{c} = 1. \tag{2.43}$$

Hence what is uncertain may or may not be real, may or may not be true or rational, whilst the real may or may not be certain. Yet only that which is certain is realized as true, rational, and real. But as Schelling maintained, the real is rational and true whether it be realized as a certainty or not. This cannot be said; although it seems to be the essence of his philosophy. But the true must be realized in some manner or another, though certainty be not essential as an element of realization.

§ 2. *Ontology and Epistemology*

If p = a phenomenon, n = a noumenon, then in general

$$(p+\bar{p}).(n+\bar{n}) = 1. \tag{2.44}$$

Now if everything is either a phenomenon or a noumenon, i.e. $\bar{p}.\bar{n} = 0$; then it follows that

$$(p.n+p.\bar{n}+\bar{p}.n) = 1. \tag{2.45}$$

If T = Truth, then, since $T+\bar{T} = 1$,

$$(p.n+p.\bar{n}+\bar{p}.n).(T+\bar{T}) = 1. \tag{2.46}$$

But if as is supposed all noumena are true, then $n.\bar{T} = 0$; and therefore

$$T.(p.n+p.\bar{n}+\bar{p}.n)+\bar{T}.p.\bar{n} = 1. \qquad (2.47)$$

Truth therefore may or may not be a phenomenon alone or a noumenon alone. If it is a noumenon, it may or may not be a phenomenon. Whereas if it is a phenomenon, it may or may not be a noumenon. Hence it is possible for it to be both a phenomenon and a noumenon. But anything that is not true cannot be a noumenon, and consequently since it must be one or the other, from (2.45) it must be a phenomenon. Unless there is no such thing as error.

However, strictly speaking the alternative between noumena and phenomena applies only to what is true, so that $T.\bar{p}.\bar{n} = 0$; and since as before, all noumena are *ex hypothesi* true, then we have

$$T.(p.n+p.\bar{n}+\bar{p}.n)+\bar{T}\bar{n}.(p+\bar{p}) = 1. \qquad (2.48)$$

Consequently a thing may or may not be a phenomenon, if it is not true. But if it is true then must it be either a phenomenon or a noumenon; and in the absence of further considerations it might be both.

Kant maintains that we have no knowledge of noumena; that all our knowledge is derived from experience and consists of phenomena, so that if we know truth at all it must be as a phenomenon. But on his own showing all phenomena are self-contradictory, and therefore not strictly speaking true. Hence we really cannot have any knowledge of the truth at all. Whether Kant himself would have repudiated this conclusion, it necessarily follows from his insistence that we can have no knowledge of noumena, that all our knowledge consists of phenomena, and that these are not true because self-contradictory. The true is in fact for Kant the thing-in-itself and can never be known. It can only be regarded as a 'regulative maxim'. His 'transcendental ideas' are, if noumena, unknown and so far as the human mind is concerned unknowable—not of course in Herbert Spencer's sense—whereas if they were knowable it would be only through the understanding by the medium of the categories; subject to the antinomies and therefore self-contradictory. This sounds like a ju-jitsu performance, as though Kant were thrown up into the air and caught from behind under the leg and thrown upon the ground again standing on his feet, against his own wish.

Hegel, however, insists that either there is no such thing as a noumenon at all, or else the true is a noumenon; so that a phenomenon which is true is a noumenon. According to this there is continuity between truth and error, between noumena and phenomena.

If $n = 0$, i.e. if there is no such thing as a noumenon, then

$$\bar{n}.\{Tp+\bar{T}(p+\bar{p})\} = 1; \qquad (2.49)$$

so that the true is necessarily a phenomenon, although of course phenomena are not necessarily true.

On the other hand, if the true is a noumenon, then $T.\bar{n} = 0$; and therefore

$$\{T.n(p+\bar{p})+\bar{T}.\bar{n}.(p+\bar{p})\} = 1; \qquad (2.50)$$

and if truth is a phenomenon then $T.\bar{p} = 0$; and we have

$$p.(T.n+\bar{T}.\bar{n})+\bar{T}.\bar{n}.\bar{p} = 1. \qquad (2.51)$$

There is thus a connecting link between truth and error through the class of perceptions called phenomena.

So far, however, since the reasoning is 'static', there can be no continuity between truth and error. We must suppose that phenomena have all some element of truth, and yet something of error, except in the limits when they are absolutely true or untrue.

This conclusion, though admirably suited to the casuist, will hardly satisfy the strict Unitarian. Yet in the light of 'dynamic' thought of practical life it is difficult to see how these results could be avoided.

The syntheses from higher to higher will ultimately lead to the truth devoid of error, according to Hegel's idea of the dialectic, but obviously only by a continuously progressive process. As we have already pointed out, this is only raising the old difficulty in a new form. For if the 'jump' from the true to the untrue is apparently avoided it is only by assuming that all error partakes of truth, and vice versa. (See p. 50.)

(A) If we represent by i = an idea,
T = the Truth,
r = the Rational realized,
R = the Real,

then since (1) Every idea is real, whether true or untrue, rational or irrational, we have $i.\bar{R} = 0$. Further, the true must be rational as it must also be an idea; hence (2) $T.\bar{r} = 0$; and (3) $T.\bar{\imath} = 0$. And it will hardly be disputed that the rational is real if realized, i.e. (4) $r.\bar{R} = 0$; (5) $r\bar{\imath} = 0$.

It follows that

$$i.R.\{(T+\bar{T}).r+\bar{T}.\bar{r}\}+\bar{\imath}.T\{(R+\bar{R}).\bar{r}\} = 1. \qquad (2.52)$$

The real therefore may or may not be an idea. If it is an idea it may or may not be rational. But if it is rational it may nevertheless not be true; although if it is true it must be rational. If it is irrational

it cannot be true. On the other hand, if it is not an idea it cannot be true or rational; whereas what is not an idea, if not real must needs be both irrational and untrue. Furthermore anything that is not an idea is irrational and untrue, whether real or unreal.

Now if the unreal does not enter into the question, it may for all purposes be eliminated from the universe of discourse. Hence we may put $\bar{\imath}.\bar{T}.\bar{R}.\bar{r} = 0$. Consequently everything is real for our purpose. The non-real does not exist for our purpose. We may assume that not only the real exists, but that everything that exists is real. Hence

$$R.[i.\{(T+\bar{T}).r+\bar{T}.\bar{r}\}+\bar{\imath}.\bar{T}] = 1. \qquad (2.53)$$

If, however, we maintain with Hegel and all idealists that the real is an idea, so that the real that is not an idea does not exist, i.e. $R.\bar{\imath} = 0$, then we have

$$R.i.\{(T+\bar{T}).r+\bar{T}.\bar{r}\} = 1. \qquad (2.54)$$

The idealistic link of reality connects the true with the untrue; the rational with the irrational.

We can now combine this equation with that previously obtained:

$$p.(T.n+\bar{T}.\bar{n})+\bar{T}.\bar{n}.\bar{p} = 1, \qquad (2.55)$$

in which truth is identified with the noumenon, whilst at the same time it is admitted to be a phenomenon; and accordingly we obtain that

$$R.i.\{T.p.n.r+\bar{T}(r+\bar{r}).\bar{n}.(p+\bar{p})\} = 1. \qquad (2.56)$$

Thus all that is rational must now be accepted as true. So far it has not been shown that an idea though real must be a noumenon or even a phenomenon. Irrational and untrue ideas which are not being perceived as phenomena, and which from what has been said are not noumena, may nevertheless exist as potentialities of the subconscious, possibly of the imagination.

(B) The general problem of metaphysics then may be stated thus. If by i = an idea,
q = a quality,
q_1 = a thing having qualities,
s_1 = a sense datum,
s = a subject of a proposition,
p = a predicate,
then in general

$$(i+\bar{\imath}).(q+\bar{q}).(q_1+\bar{q}_1).(s_1+\bar{s}_1).(s+\bar{s}).(p+\bar{p}) = 1. \qquad (2.57)$$

Now obviously (1) $i.\bar{q} = 0$ and $q.\bar{\imath} = 0$, i.e. $i = q$, that is qualities and ideas are identical; for all qualities are necessarily ideas, and all ideas are qualities of some other idea.

Similarly (2) every predicate is a quality, and every quality is a predicate.

$$p.\bar{q} = 0 \text{ and } q.\bar{p} = 0, \text{ i.e. } q = p. \quad \therefore p = q = i. \qquad (2.58)$$

Now everything with qualities is a sense datum, and vice versa. $\therefore$ (3) $s_1 = q_1$.

We may, therefore, before proceeding further, write the general equation thus:

$$(i.p.q+\bar{\imath}.\bar{p}.\bar{q}).(q_1s_1+\bar{q}_1\bar{s}_1).(s+\bar{s}) = 1. \qquad (2.59)$$

Again, we may further assert that (4) $i.\bar{q}_1 = 0$, i.e. every idea has qualities. Also that (5) $s.\bar{\imath} = 0$, i.e. a subject of a proposition is an idea. And again (6) $q_1.i.\bar{s} = 0$; anything with qualities that is an idea is a subject of a proposition.

Hence we readily obtain that

$$i.q.q_1.s_1.s.p+\bar{\imath}.\bar{q}.\bar{s}.\bar{p}.(q_1s_1+\bar{q}_1\bar{s}_1) = 1. \qquad (2.60)$$

Consequently a thing with qualities or sense datum is not necessarily so far as the premisses go an idea. The whole theory of idealism from Berkeley onwards is thus excluded.

If, however, it be affirmed that $q_1.\bar{q} = 0$, i.e. that whatever has qualities is itself a quality of something else, i.e. every class includes another class, the equation represents an infinite series: for the thing with qualities being a quality of something else, which according to the equation is a quality of something else, and so on *ad infinitum*. Hence

$$iqq_1s_1sp+\bar{\imath}\bar{q}\bar{s}\bar{p}.(\bar{q}_1\bar{s}_1) = 1. \qquad (2.61)$$

Here, of course, there is no sense datum that is not an idea. The only alternative to an idea is something which has no qualities whatever.

Now since everything must have some quality of some sort, (7) $\bar{q}_1 = 0$. Hence the term of complete negation corresponding to Herbert Spencer's Unknowable must be omitted. If $q_1\bar{q} \neq 0$ we therefore get

$$q_1s_1(iqsp+\bar{\imath}\bar{q}\bar{s}\bar{p}) = 1. \qquad (2.62)$$

Thus it is still possible that a thing may have qualities and be a sense datum without being an idea. Only if the series $q_1.\bar{q} = 0$ holds does idealism necessarily follow. This in fact is equivalent to Kant's conclusion; for the series only holds owing to the concepts of the understanding. But if we admit the validity of the understanding, then $q_1s_1iqsp = 1$. Q.E.D.

Thus not only are subject and predicate one but they become identified with the thing-in-itself, which is going beyond Kant.

The precise interpretation of this equation is exactly the battle-ground of idealistic philosophers at the present day. McTaggart maintained that each member of the class thus represented is an individual separate and distinct from every other. Each is its own subject and object. But he denied that the Whole could be regarded as having a subject and object of its own. There is no reason, however, so far as I can see, why there should not be a subject of which all the predicates could be predicated without its being itself identical with any one of the other subjects. In this respect, then, the one subject is the infinite personality which is in no way to be identified with the other entities which remain independent and distinct. Strictly speaking, the Whole is not and cannot be self-conscious. But the infinite self-consciousness which perceives the Whole is not incompatible with the separate existence of the individual parts, as self-conscious units of reality. This in fact was Lotze's view of the question.

Hence $\sum s.p+S_0\sum p = 1$, or putting $X_0 = S_0.\sum p_m$, we may write the equation thus:

$$X_0+\sum_1^n s_m.p_m = 1, \text{ or } S_0.P_0+\sum_1^n s_m.p_m = 1 \text{ when } P_0 = \sum_0^\infty p. \tag{2.63}$$

(C) Idealistic-Realism:

Let s = sense data.

e = things that exist.

q = qualities of anything whatever.

Then since (1) 'all sense-data are things that by their very nature must be admitted to exist', $s.\bar{e} = 0$.

Also 'Everything that exists is a quality, since it must be a predicate of something else', (2) $e.\bar{q} = 0$.

Therefore $$e.q.(s+\bar{s})+\bar{e}.\bar{s}.(q+\bar{q}) = 1. \tag{2.64}$$

Now if i = an idea, we have (3) $i.\bar{q} = 0$ and $q.\bar{\imath} = 0$,

i.e. $$q.i+\bar{q}.\bar{\imath} = 1.$$

And therefore $$e.q.i.(s+\bar{s})+\bar{e}\bar{s}(q.i+\bar{q}.\bar{\imath}) = 1. \tag{2.65}$$

If m = things in the mind, $i = v.e.m$, i.e. 'Ideas are things that exist in the mind'.

This is equivalent to $i.(1-e.m) = 0$ or to $i.\bar{e}.m = 0$; $i.e.\bar{m} = 0$; $i.\bar{e}.\bar{m} = 0$.

Therefore $$e.q.i.m.(s+\bar{s})+\bar{e}.\bar{q}.\bar{\imath}.\bar{s}.(m+\bar{m}) = 1. \tag{2.66}$$

Thus mind which exists as an idea may or may not be a sense datum. And it is possible so far as the premisses go that there might be mind *in posse* that is neither a sense datum nor an idea of any kind.

It is true, as Russell affirms, that a thing may exist without being a sense datum; but it must have qualities of some sort and then it can only be an idea, and exist in a mind as a perception.

Hartmann's unconscious mind and Clifford's 'mind-stuff' would both apparently come under this class.

If it be asserted that 'all qualities are sense data', i.e. $q.\bar{s} = 0$; then we obtain $e.q.i.m.s+\bar{e}.\bar{q}.\bar{i}(m+\bar{m})\bar{s} = 1$. (2.67)

This is obviously the Berkeleyian idealism, i.e. everything that exists is an idea and a sense datum.*

(D) According to Kant in the *Critique of Pure Reason*: 'All knowledge begins with experience, but does not arise with it.' This proposition he held cannot be doubted. But that which cannot be doubted need not on that account be what he means by knowledge. Moreover it may not be derived from experience; and even if it were it need not begin with it, or even from it. Hence the proposition does not in any way imply that what cannot be doubted is necessarily knowledge, such as faith or belief that is not based on experience, or transcendental ideas not derived from experience. It would be necessary to add that that which is not derived from experience is doubtful, or conversely that which cannot be doubted is necessarily knowledge derived from experience.

If we put K = knowledge; e = experience; b = things that begin with something; a = those that arise with something; and d = those that are doubtful. Then all $K = Kbe\bar{a}$ and $Kbe\bar{a}d = 0$.

$$\therefore \quad Kb\bar{a}e\bar{d}+\bar{K}(b+\bar{b})(a+\bar{a})(e+\bar{e})(d+\bar{d}) = 1. \qquad (2.68)$$

Unless therefore it be implied that that which can be doubted either arises or begins with experience and as such must be knowledge; or whether based on experience or not must be knowledge (thereby excluding all intuitive and instinctive, emotional and transcendental ideas, actual or possible), Kant's premisses as given above would not exclude the possibility of certainty without actual knowledge, as in the case of faith or other form of expectation or belief. The principle of induction is of this nature. So is the belief in the law of causation and the continuity of Nature. They are not knowledge whether based upon experience or not; and whether they do or do not begin or arise in it. If it be admitted only with actual experience is certainty possible, i.e. $\bar{d}\bar{e} = 0$, and all that is in experience is knowledge, $e\bar{K} = 0$, then the second term in the equation vanishes, so far as $\bar{d}$ is concerned; and all that is not

* Mind may not always be an idea but only a potentiality. It would not necessarily exist then except potentially. It is here that 'emergence' may arise.

knowledge whether beginning with experience, or not arising in it, is doubtful, $Kb\bar{a}e\bar{d}+\bar{K}(b+\bar{b})(a+\bar{a})\bar{e}d = 1.$ (2.69)

(E) The following analysis of Kant's theory of the relation of time to substance is worth considering. If t = time, τ = what is in time, and s = substance, p = permanent, u = a unity. Then he says that 'time is a unity but not anything permanent, for it necessarily consists of change'. 'Substance, however, is something permanent, and anything permanent is a substance *ex definitio*.' Hence

$$t\bar{u} = 0; \quad tp = 0; \quad s\bar{p} = 0; \quad p\bar{s} = 0; \quad \text{or } s = p. \quad (2.70)$$

Consequently

$$tu\bar{s}\bar{p}(\tau+\bar{\tau})+\bar{t}(u+\bar{u})(sp+\bar{s}\bar{p})(\tau+\bar{\tau}) = 1. \quad (2.71)$$

Therefore time is not a substance. And if there be anything substantial and permanent it is not time; it may or may not be a unity; and may or may not be in time.

The very idea of substance thus implies something transcendental that may not be time. For instance, if the soul be a simple substance it must be something permanent. There is nothing in the above remarks to suggest that the soul though a unity and not in time is a substance, or in fact permanent. If it be affirmed that that which is not in time must be permanent, $\bar{\tau}\bar{p} = 0$, then the unity which is not in time must be a substance and permanent.

This interesting inference might well have been overlooked by the ordinary logic, though it obviously is fully in accordance with it. We cannot, however, say whether that which is not in time can or cannot be permanent. The expression would appear to have no meaning. But upon this point enough perhaps has already been said, since timeless existence cannot logically be excluded in discussing the transcendental ideas which are presumably beyond the forms of sensuous intuitions of space and time. Here, indeed, we have one more illustration if such were necessary of the superiority of the analytic method adopted in this book.

(F) *Kant's supposed refutation of Idealism* (vide *Critique of Pure Reason*, Meiklejohn, p. 167).

Let I = the Ego.

c = something conscious of its own existence.

x = things determined by something.

p = things presupposing existence of something permanent in perception.

i = something within the Ego.

We have then that

$$\left.\begin{array}{l}(1)\ I\bar{c} = 0.\\(2)\ I\bar{x} = 0.\\(3)\ x\bar{p} = 0.\end{array}\right\} \qquad (2.73)$$

Therefore from (2) and (3) we obtain that

$$x(I+\bar{I})+\bar{x}\bar{I} = 1, \text{ and } xp+\bar{x}(p+\bar{p}) = 1.$$

Therefore
$$(I+\bar{I}).px+\bar{x}\bar{I}(p+\bar{p}) = 1. \qquad (2.74)$$

Now Kant assumes (4) that $ip = 0$; (5) $i\bar{I} = 0$; and (6) $i\bar{x} = 0$.

Therefore
$$\bar{i}\{(I+\bar{I}).px+\bar{x}\bar{I}(p+\bar{p})\} = 1. \qquad (2.75)$$

The assumption (4) is clearly a begging of the question, i.e. Kant virtually implies that the internal and permanent exclude each other, because the internal excludes something which the permanent necessarily includes or implies. If y be that something, then either $iy = 0$, and $p\bar{y} = 0$, or else $i\bar{y} = 0$, and $py = 0$; and therefore $ip = 0$, by eliminating y.

There is no reason whatever to suppose that what is in me cannot presuppose the existence of something permanent in perception. In other words that any such idea or class of ideas separates the two classes of ideas i and p. Kant presupposes a contradiction where there may be none.*

§ 3

Anselm's Ontological Proof.

The actual difficulty in the problem before us can be traced to St. Anselm, Archbishop of Canterbury, the earliest and one of the greatest of the Schoolmen. His words were: 'Certe id quo maius cogitari nequit, non potest esse in intellectu solo. Si enim vel in solo intellectu est, potest cogitari esse *et in re*; quod maius est. Si ergo id quo maius cogitari non potest, est in solo intellectu; id ipsum quo maius cogitari non potest, est quo maius cogitari potest. Sed certe hoc esse non potest.'

Literally this means: 'Certainly that, than which nothing greater can be thought, cannot be in the intellect alone. For even if it is in the intellect alone, it can also be thought to exist in fact and that is greater. If then that, than which nothing greater can be thought, is in the intellect alone; then the very thing, which is greater than anything which can be thought, can be exceeded in thought. But surely this is impossible.'

* The Universe without me includes the external world. The Universe within me includes as a world of ideas truth itself, as permanent and independent of me, it would appear, as the external world.

If x = 'that, than which nothing greater can be thought'.
y = 'that, which is in the intellect alone'.
z = 'that, which can be thought to exist in fact'.
w = 'that, which can be exceeded in thought'.

Then the premisses are (1) 'If $x\bar{y} = 0$, then $x\bar{z} = 0$', i.e. $xy\bar{z} = 0$; (2) $z\bar{w} = 0$, and (3) $xw = 0$.

Therefore, since $(x+\bar{x})(y+\bar{y})(z+\bar{z})(w+\bar{w}) = 1$,

it follows that $x\bar{y}\bar{z}\bar{w}+\bar{x}(y+\bar{y}).\{zw+\bar{z}(w+\bar{w})\} = 1$.

Hence $$xy = 0; \quad xz = 0; \quad \bar{x}z\bar{w} = 0. \tag{2.76}$$

Anselm's inference was: 'That, than which nothing greater can be thought, is not in the intellect alone.' But it also follows that 'it cannot be thought to exist in fact; and it cannot be exceeded in thought', since $x = x\bar{y}\bar{z}\bar{w}$.

If, on the other hand, we interpret his premisses to mean that (1) $xy\bar{z} = 0$; (2) $xyz\bar{w} = 0$; (3) $xw = 0$. Then clearly

$$x\bar{y}(z+\bar{z})\bar{w}+\bar{x}(y+\bar{y})(z+\bar{z})(w+\bar{w}) = 1. \tag{2.77}$$

Hence, that, than which nothing greater can be thought, cannot be in the intellect alone, nor can it be exceeded in thought; but it may or may not be thought to exist as a fact. If it must be thought of as existing as a fact, then must $x\bar{y}\bar{z} = 0$. So that it is still not in the intellect alone, though something that can be thought to exist as a fact.

It may of course be said that $\bar{y}\bar{z} = 0$, i.e. that that which is not in the intellect alone, if it exists, must be thought of to exist in fact. And that, than which no greater can be thought, not being in the intellect, but from the nature of perfection inconceivable except as existing, it follows that it can be thought of to exist in fact. In other words $x\bar{y}\bar{z} = 0$. Consequently there remains no doubt but that $x\bar{y}z\bar{w}+\bar{x} \ldots = 1$. (2.78)

Hence it is essential that $z\bar{w} \neq 0$, that is that which can be thought to exist in fact cannot be exceeded in thought; contrary to the assumption in the (2) premiss of the first interpretation. It is obvious that this is the correct interpretation of Anselm's view.

Anselm thus with good reason defines that only to be perfect which exists not merely in a subjective, but also in an objective mode.

'It does no good', says Hegel, 'to put on airs against the Ontological proof, as it is called, and against Anselm thus defining the Perfect. The argument is one latent in every unsophisticated mind, and it recurs in every philosophy, even against its wish and without its knowledge—as may be seen in the theory of immediate belief.'

The finite does not involve the idea of existence, but the infinite does. In short the conception and existence of the finite is set in antagonism to the infinite. For

> 'the finite possesses objectivity of such a kind as is at once incongruous with and different from the end or aim, its essence and notion. Or, the finite is such a conception and in such a way subjective, that it does not involve existence. This objection and this antithesis are got over, only by showing the finite to be untrue and these categories in their separation to be inadequate and null. . . . To call a thing finite, means that its objective existence is not in harmony with the thought of it. . . . The *unity* which is enunciated as the *supreme perfection* or, it may be, *subjectively* as the *true knowledge* is *pre-supposed*, i.e. it is assumed only as *potential*. . . . The identity, abstract as it thus appears, between the two categories may be at once met and opposed by their diversity; and this was the very answer given to Anselm long ago. . . . The identity is thus seen to be one into which they spontaneously pass over, and in which they are reconciled.'

This idea runs throughout the whole of Hegel's logic of the notion. He thus fully supports Anselm, though going beyond him: for he holds that the finite and infinite merge into one; in the higher synthesis of the Absolute. Here, however, we are brought back once more to the fundamental problem that, as we have seen, Hegel has not really solved. The categories conflict in the understanding, but are transcended in the pure reason. The contradiction of the antinomy can only be overcome by recognizing that than which nothing greater can be thought, to be not in the intellect alone, but something which can be thought to exist as a fact, and not exceeded in thought, other than that than which nothing greater than it can be thought. Thus Anselm briefly anticipated not only Descartes, but Kant and Hegel, in the fundamental issue. The notion unites the irreconcilable. It is the *thing per se*; the Absolute, without which all is meaningless and purposeless.

Man alone of all God's creatures has the capacity to conceive the being and essence of the Absolute as the very notion and the basis of reality, of knowledge and of truth: the unity of the universal, the particular, and the singular, or individual constituting a relationship between them; such as identity for the universal; difference for the particular; and opposition for the singular. The universal, however, is not only an identity, but at once a difference or concrete idea different from the particular and singular; and further also opposed or contradictory to them; in this opposition, however, it is identical with them and their true ground on which they are uplifted. A similar remark applies both to the particular and the singular which are reciprocally related in the totality, as the whole of which they constitute the parts.

The difference in identity; the opposition and agreement; the discontinuity and continuity are manifest only in the understanding as the result of the development of the notion through the medium, as it were, of the categories. The emergence is the outcome of the expansion of the function containing real and imaginary terms such as $(a+b\sqrt{-1})$ as a potential, or possible, reality to which the category of actual existence cannot be applied, until it is expanded through the aid of some other idea or quality by whose means, or through the combination with which it is brought into being so as to constitute an actual part of the totality. This is the nature of *emergence*.

§ 4

The roots of the equation $x^n = x$ (p. 199) afford the explanation of this most mysterious paradox. And although Hegel in the *Wissenschaft der Logik* (vol. ii, pp. 258–9) insists that neither the geometrical nor algebraical representation is adequate to give the full meaning and explanation of the difficulty, he did so because he regarded these sciences, as they were regarded at the time, as merely applicable to quantity, and had no notion of their application to logic. For the conception of class is fully expressible by algebraic symbols and rules as well as by the geometry of non-Euclidean space, to which algebraic treatment extends as well as to the more ordinary conceptions of logic, as a particular branch. This is what Leibnitz aimed at and Boole so ably proved to be a feasible process. The correct interpretation of Boole's methods and results carries one far beyond the immediate scope and purpose of his work. But as we have seen it affords an elegant explanation, not only of the laws of thought in the Aristotelian sense, in which Boole himself regarded them, but further of the Hegelian Dialectic, leading inevitably as it does to the full and thorough explanation of the problem of the one and the many, and, in the highest sense, to the Platonic universal as the ultimate manifestation of truth and reality.

In the judgement as expressed by a proposition is this unity realized. No amount of probability or approximation to the truth will be sufficient to render a judgement possible, so far as an apprehension of the truth may be concerned, unless this intuitive power of apprehension of the unity in diversity of the subject and predicate is present. For this reason k must be assumed to be $= 1$ (see Chapter XIII, p. 226), wherever certainty is possible. Even the Pragmatist will admit this. And whatever the views he held upon other matters of a controversial nature, there can be no

doubt as to the actual value of this intuition in ordinary questions upon which a judgement may with competence be formed. Without it there may be confidence and expectation, but there can be no certitude.

If A = the Absolute, $A = 1$, and $\bar{A} = 0$. And we may write, where A has the qualities, let us say, $x_1x_2\bar{x}_3x_4\bar{x}_5x_6 \ldots x_n$,

$$A.x_1x_2\bar{x}_3x_4\bar{x}_5x_6 \ldots x_n + \bar{A}(x_1+\bar{x}_1)(x_2+\bar{x}_2)(x_3+\bar{x}_3) \ldots (x+\bar{x}_n) = 1. \tag{2.79}$$

Hence, since that which is not Absolute is $= 0$, it follows that the contradictory attributes as in the case of Being and non-Being may be applied to it. Every non-A is and is not

$$x_1, x_2, x_3, \ldots x_n, \quad \text{i.e. } \bar{A}\bar{x}_1 = 0, \text{ and } \bar{A}x_1 = 0, \text{ \&c.}$$

There can be no gradation from A to $\bar{A}$. The Absolute either is, or is not. This is the peculiarity of the Hegelian philosophy. The Dialectic clearly applies only to the unreal, i.e. to appearance.

Yet if the Absolute has qualities, it cannot be said to be any one of them as distinct from the others. Although

$$A = x_1 = x_2 = x_3 = \ldots x_n = 1; \quad \text{i.e. } A^{n+1} = 1. \tag{2.80}$$

The qualities themselves are not the Absolute, but only the whole combined can be said to constitute the Absolute reality.

No amount of synthesis can enable us to attain to this. The capacity to perceive reality must be present. Approximations will not do. The barrier is insuperable. And only cognitive faculty affords the power of cognition and of recognition in the ultimate perception of truth as the veritable reality.

Since the Absolute cannot be contradictory, it must be an identity. Time and space, as we have seen, belong to appearance; but cognition of what is absolutely true must needs be absolute; and as such is entirely independent of both space and time. That which is absolute is eternal and above the whims and scorns of time and the limitations of space; or the properties of matter, some of whose properties it may share, though only singly without contradiction.

§ 5

The essential and peculiar point in the philosophy of St. Thomas, on the other hand, as the Rev. M. C. D'Arcy, S.J., has pointed out in his paper on 'Knowledge according to Aquinas' (*Proceedings of the Aristotelian Society*, 1927–8), is the two-sided nature of the known object. It is in itself and yet related to the subject. His theory is an attempt to reconcile these two views. It thus appears to make

noumenon and phenomenon one, while maintaining the independent existence of the noumenon, as we might put it in more modern terms. The object thus acquires a dual nature. The subject knows the object itself, and not an intermediary or image or derivative or *a priori* form or mental construction but the thing itself; whilst this latter is independent of the knower.

'Truth is the conformity of the mind with reality and this conformity consists in the correspondence of the thought with the real object. Reality for Aquinas consists both of what exists and what is possible. Knowledge is of reality and direct. Nevertheless the correspondence theory shows that he believed in concepts or ideas as intermediaries which he calls species or forms. In the direct act of knowing we are conscious only of the real object. But in reflexion we become aware of the medium which is the form. If the mind is capable of knowing truth then it knows reality, the world of what is. Truth, however, does exist. We know the world as it is and therefore truth is both possible and actual.'

That truth exists is self-evident because, whosoever denies truth to exist concedes that there is no truth; for if truth does not exist, it is true that it does not exist; but now if there is something true, there must be some truth; for doubt is an affirmation.* If 'our intellectual faculties know only the impression made upon them', then science would be impossible, for one man might say that Socrates was a wise man, and another that he was a fool, and both might be right, or both be wrong.

The concept does not set the mind in immediate contact with the individual object. But if an experience be given in which the real individual is attained, then the difficulty disappears; such an experience is attained or given by self-knowledge. The subject is itself a thinking, reflective, concrete, individual existing reality. 'Once existence or individual being as existent is reached then the intellect is on *terra firma*.' Here at least we have an unmistakable instance of the 'self-sufficiency of knowledge': where mind conforms with reality, thought with thing. 'The truth then is that the object known is two-sided. In knowledge we attain the object as it is in itself; that is one side. But when we lay stress on the word known, then the object cannot be the single, external and unmultipliable thing because it belongs to a different order, the order of thought. . . .' Knowledge, as Cajetan remarks, is a perfection. 'The coincidence of the knower and the known, of the subject and

* This is a good example of the Russell contradiction and applies also to the basis of Descartes' *Cogito ergo sum*. I do not think it can be explained, or explained away, by the theory of Types, as given by Russell and Whitehead in *Principia Mathematica*. The coincidence, coexistence, or emergence of two ideas of different types is what constitutes a relationship between them. Bradley lays emphasis on this.

the object in *the identity of one act*; here is the whole metaphysical secret of knowledge as such.'

If S, O, n, and p stand for the subject, object, noumenon, and phenomenon, respectively, then since in general,

$$(S+\bar{S})\,(O+\bar{O})\,(n+\bar{n})\,(p+\bar{p}) = 1, \qquad (2.81)$$

and the subject is an object, $S\bar{O} = 0$; but everything is either a subject or an object, $\bar{S}\bar{O} = 0$; and the object is a noumenon, $O\bar{n} = 0$.

Then $$(S+\bar{S}).O.n.(p+\bar{p}) = 1. \qquad (2.82)$$

The phenomenon is therefore an object; but the object may or may not be a phenomenon, just as it may or may not be a subject.

In the case of the Ego, it is both subject and phenomenon at once. That is, if E stand for the Ego, $E\bar{S} = 0$, and $E\bar{p} = 0$.

Hence $$ES.O.n.p+\bar{E}(S+\bar{S}).O.n.(p+\bar{p}) = 1. \qquad (2.83)$$

The Ego therefore is subject and object, phenomenon and noumenon combined; while the non-Ego though object and noumenon is not necessarily a subject or phenomenon. In the Hegelian philosophy the object and subject are one, so that $O\bar{S} = 0$; and presumably the subject is a phenomenon or nothing, therefore

$$(E+\bar{E}).S.O.n.p = 1. \qquad (2.84)$$

The independent existence of the object together with its identity with the subject constitutes one of the striking peculiarities of the Thomist's position, distinguishing it from modern idealism as partaking of realism. But the difference though marked deserves careful examination, as the sequel will, it is hoped, make clear.

The philosophy of St. Thomas Aquinas admits of being analysed and expounded by the methods we have proposed. It happens, however, to be largely a development of the Aristotelian philosophy. Not in the sense that Hegel's was Parmenides', or Bergson's, Heraclitus'; but something far more intimate, and much closer in outline as in detail. For, notwithstanding some vague anticipations on the part of Parmenides, Hegel was perhaps the most original thinker since Plato. But Aquinas was professedly a follower and expounder of Aristotle. We meet with the same difficulties again and again, in the endeavour to tackle the same questions of 'the one and the many'; 'the continuity of genera and species'; and 'the unity of contraries'.*

* An excellent exposition of this particular point in the philosophy of St. Thomas is given by Dr. Stefano Bersani in his *Immaterialità e Pensiero* (Francesco Ferrari, Rome, 1922).

Bersani regards the whole of the Thomist's philosophy as based upon the principle 'Quod potest cognoscere aliqua, oportet ut nihil eorum habeat in sua natura'. This is based upon the opposition of contraries and is known as the Q.p.c. principle.

If a = certain qualities,
x = that which perceives a,
y = that which has a as a quality,
z = any particular thing that exists as a reality,
then if z is x, and z is not y; i.e. if $z\bar{x} = 0$, and $zy = 0$, then $zxy = 0$. That is the three ideas are incompatible, but any two or each one separately is possible, so far as the principle stated above may be concerned.

Thus, $(x+\bar{x})\,(y+\bar{y})\,(z+\bar{z}) = 1$ in general.

But $$z.x.y = 0. \tag{2.85}$$

$$\left.\begin{aligned} \therefore\quad & x(y\bar{z}+\bar{y}z+\bar{y}\bar{z})+\bar{x}(y+\bar{y})\,(z+\bar{z}) = 1, \\ \text{or,}\quad & y(z\bar{x}+\bar{z}x+\bar{z}\bar{x})+\bar{y}(z+\bar{z})\,(x+\bar{x}) = 1, \\ \text{or,}\quad & z(x\bar{y}+\bar{x}y+\bar{x}\bar{y})+\bar{z}(x+\bar{x})\,(y+\bar{y}) = 1. \end{aligned}\right\} \tag{2.86}$$

Hence if a thing is not real, the contradiction becomes possible between the opposites x and y. Although this follows also from the Aristotelian logic, it is not emphasized and insisted upon as it is here, but only tacitly assumed that the cognizant is real, i.e. $x\bar{z} = 0$, which is not necessarily implied in the general principle $zxy = 0$.

If $x\bar{z} = 0$, then $x\bar{y}z+\bar{x}(y+\bar{y})\,(z+\bar{z}) = 1$.

Therefore $xy = 0$ under all circumstances whatever.

This is the dualism of the Thomist and Aristotelian philosophies, the supreme difficulty of reconciling the irreconcilable. The two extremes are present with relentless incompatibility; like the difference between subject and predicate, already discussed. The two can be reconciled only in the transcendental ideas; like circles that pass through two points at infinity. All classes do likewise overlap. There is a common factor that unites the conflicting elements and moulds them into one, how different soever they might be, and gives them a living connexion as an identity in the highest reality. This is the transcendental world of spiritual reality, without which all is chaos and unmeaning. The reality does not admit of contradiction and therefore must be identical with itself. This identity is truth. It is the unity of the one in the many, and the many in one. All the imaginary roots of unity in the identical relation $x = x^n$ (p. 199) yield a contradiction; only the one real root, *one*, gives the true and consistent result.

Like the curvature of space through which the infinite returns into itself, so the continuity of genera and species leading to opposite extremes ultimately ends in the unity of the contradictory elements of which truth is composed: as in the simple judgement S is P, where S is identical with some P, and yet something more than mere P; while P is not necessarily S. The 'curvature' of thought, by which every idea as a notion, returns into itself in the infinite association or chain of ideas. A point is the meeting-place of opposites, as a judgement is the unification of two contradictory elements of thought. Only by this means is consciousness rendered possible. But the Q.p.c. principle of St. Thomas implies an absolute irreconcilability between the two substances of matter and spirit.* The subtlety of the Angelic Doctor's ideas in grappling with this difficult problem is very much surpassed by modern conceptions of the notion. But it is to his credit that he perceived with the utmost clearness the spiritual nature of man, though he failed through the Aristotelian dualism to grasp the key to the difficulty in the problem. St. Thomas marks a step, and that a very important step in the right direction, towards the solution of the difficulty. It is not to be supposed that he arrived at the ultimate solution of the problem, or said the last word possible upon the subject, far from it.

Like the production of water from hydrogen and oxygen, the theory of emergence explodes the Q.p.c. of Thomistism, just as $\sqrt{-1} \times \sqrt{-1} = -1$. Life and Mind can emerge from inanimate senseless matter. But the manner is a question of logic.

§ 6

If in general x is not y, i.e. $xy = 0$, it is nevertheless possible that $x^m = y^n$; in which case $m.\log_e x = n.\log_e y$, where if $e^p = x$ and $e^q = y$, i.e. $m.p = n.q$. This is the exponential transformation in the psychological connexion between sensation and stimulus, leading to Weber's Law.

Hegel regards the notion as a repetition of the identity. But only three or four times; obtained first through negation; then 'reflexion'; and later negation of the negation. The algebraic treatment of the subject, however, makes it clear that there might be any number of such 'reflexions' of an object into itself. But a particular number with one might coincide with an entirely different number in another, provided that they were exponentially related.

* See also p. 124, footnote.

Thus then and then only is there continuity between x and y; and these two classes might be otherwise discontinuous. It will be asked, How can the mth 'reflexion' of x coincide with the nth 'reflexion' of y? The best physical illustration is in the phenomenon of interference of light. Two rays of light from the same source, which have been reflected a different number of times from a number of mirrors, can, if made to meet at a given point, actually interfere. Now though the analogy is not perfect it can be seen that two events can coincide in time under such similar circumstances.

This can obviously be expressed thus:

$$e^{ipm} = e^{iqn} = e^{ir} = \cos(ipm) - i\sin(ipm)$$
$$= \cos(iqn) - i\sin(iqn) = \cos(ir) - i\sin(ir).$$

The identities here become coincidences; and we can arrive at a theory of emergence by coincidence or coexistence in thought; but far exceeding the Hegelian conception of the notion in its generality and applicability. Hegel no doubt recognized, and was the first I think to recognize, the differences in identity. That $x.x$ is not strictly the same as x. But though he carries this principle to three or four identities, he does not perceive that it applies to $x^m = x$, or $y^n = y$. Where these coincide, then and then only is there complete identity between x and y. And x is otherwise not y.

This mergence of two ideas constitutes an emergence possessing continuity between the two original ideas. It is what Hegel failed to express. But how, it will be asked, is the jump from appearance to reality to be made by such means? It is not made by such means, except when $x = 1$. The continuity in all other cases is based upon the imaginary roots of unity, and is likewise imaginary, illusory, apparent, and not real.

Like two parallel lines, however, the difference may merge into an identity in the infinite. That is if m and n are infinite, then $x^m = y^n$. But we have seen that this is possible numerically if m and n are finite when $m/n = \log_e y/\log_e x$. The interpretation of the logarithm of a class in the calculus of logic must therefore mean the number of times that it—the class—is identified with itself; as in the repetition in time, when an idea ultimately coincides with another, not necessarily of the same 'degree' or 'moment' as itself. This may be what Hegel meant by 'moment' but he does not seem to have perceived that the continuity between ideas is the coincidence of ideas of different 'moments' or degrees. For instance the angle $A° + m360°$, m being an integer, is identically the same angle

as $A°$; but it has not the same trigonometrical significance. If $A° \neq B°$, yet $A°+m360° = B°+n360°$;

m and n not being both integers. Hence the event which results from m rotations of A may coincide with another depending upon n rotations of B; provided that $(n—m) = (A°-B°)/360$.

Nevertheless the two angles A and B will, so far as directions in space may be concerned, be entirely different. Their identity will refer only to the respective multiples. The conception is familiar to everybody versed in the theory of quaternions and vector analysis, so that there is no ground for complaint that it is necessarily obscurantist when applied to logic. The reproduction, though identical with the original, differs at least from it in having an interval of time between it and the original, and coming after it rather than before it. But it may differ from it in other respects whilst remaining identical. In the phenomenon of 'beats' in acoustics we find a somewhat similar effect. It is found that two notes which are approximately the same and yet different, coincide and reinforce each other at regular intervals depending upon the interval of frequency between them. The Hegelian 'reflexion' can be repeated any number of times; so that m reflexions of one idea would coincide with n reflexions of another with which it was continuous. This is the secret which only the algebraic treatment of the subject can clearly disclose.*

The whole theory of commeasurables is of this nature. And yet this is very different from the mere addition or multiplication of units. 'Das Moment' of an idea is *an empirical difference in an apparent identity*, like the difference in mass between two bodies otherwise apparently the same. 'Das Moment' is the degree or intensity of an idea, and constitutes the really difficult part of the Hegelian philosophy. But the analogy with the conception of rotation in quaternions and vector analysis is perhaps the best aid to its comprehension. For the idea which returns into itself in an identity, is never actually the same as the idea that has not been so transformed. The operation constitutes a change; and the theory of relativity affords many illustrations of a similar nature; the simplest of which is that of two observers in the same place who have travelled round the Earth in opposite directions and returned to the same spot from which they started. Their times will differ by two days, or a multiple of two days according to the number of times they have performed the process.

* The idea of group velocity in Wave-Mechanics is of this nature. The electron is supposed to be the result of the superposition of waves of different periods and velocities.

Now if the dimensions of a body gradually altered with the time it cannot be said to remain identically the same. Hence two bodies of different size, whose standards differed gradually at different rates could in the course of time be regarded as of the same size. Change of time, or change of space, as likewise change of dimensions, can give rise to an identity with difference. And the logical series of identity, such as we have been considering, can constitute a difference in identity; for the members of a class which is itself a member of itself *ad infinitum* cannot be said to be precisely the same with itself, any more than 2π is the same as $2m\pi$. And yet the class of angles $2\pi m\theta$ are all identically the same though not the same owing to the rotation.

Hence we see that two ideas can thus merge into one and become identical. This is the 'secret of Hegel' as I understand him. But just as he found fault with Kant for limiting the categories to twelve, so might we express the same wonder that in trying to arrive at the notion and the idea he should have been satisfied with half a dozen negations or 'reflexions', when in truth the number may be infinite; even if the number for the identification of two classes is not always so. The constancy *mp* (*vide* p. 144) is an empirical fact of experience, like the constancy of the 'moment of momentum' of a different nature in physics. Deeply as we may go, there is something deeper, and yet in the infinite limit it is itself. This is the very essence of the notion. Not 'essence' in Aristotle's sense, which was only substance. Nor 'essence' in Hegel's sense, which was only little more or less than 'form' or 'ground'; but essence as the very alpha and omega of reality in the notion, the soul and the life and truth of all that IS. The Idea or Λόγος.

The analogy of a spiral staircase is perhaps the best that can elucidate the gradual progression of an identity towards the Absolute. As the screw motion of a rotation ever returns into itself, as it were, yet marking progress in a different dimension, so the bare logical identity constitutes a difference according to its dimension in thought.

We are thus led to a world of potential ideas which in certain circumstances, distinct from the purely imaginary, *emerge* into the limelight of consciousness. The idea is, or rather particular ideas are, matter itself. And the properties of these must be regarded as no arbitrary conceptions, but rigorous relationships, depending upon the constitution of the monads, with the Universal Monad as the archetype or universal stamp of the reality beyond human consciousness. Science is possible because, and only because, a

universal relationship of ideas, namely truth, exists. The quest of this is the pursuit and the purpose of science no less than of philosophy. But since science confines itself to natural or material phenomena, which are limited within the confines of space and time, as distinct from the transcendental ideas of the pure reason, its sphere of investigation is bounded by these intuitions, and accordingly can be but mere appearances and self-contradictory. Hence the striking anomalies that for ever seem to divide earnest and serious thinkers into hostile camps, like those of the Darwinian theory and the spiritual nature of man. The philosophy of Kant transcends these anomalies; and in a clearer sense so does the pure idealism of Hegel, and the ideal-realism of to-day. Technically speaking, science deals with 'ideas' rather than with 'notions' in the Hegelian sense. The electron no more exists independently of my mind or the minds of other human beings than any other phenomenon; for the electron is an inferred phenomenon. It is a self-contradictory idea, like every other conception in space and time; a unit and yet possibly divisible like the atom itself, a mere contradiction as a unit and therefore an appearance or illusion. The electron, and not only the electron but the quantum has vanished not 'into air, into thin air', but like the nimbus itself into the non-existent, as an entity; taking away with it nothing but energy in its annihilation. It is now regarded as a group-wave. But mind which seemed to emerge from matter in space and time is found on careful analysis to transcend them in the world of the supersensual.

CHAPTER VII

SOME ANTINOMIES

§ 1

BEFORE proceeding to a full analysis of Hegelian Dialectic we may examine further some more or less familiar arguments of a pre-Hegelian type, many of a fallacious nature, or antinomies. The ontological argument has been applied to the immortality and freedom of the Ego. Kant endeavoured to refute it in both cases as previously in regard to the existence of a Perfect Being, on the ground that although otherwise unanswerable it is in each case based upon an analytical proposition. Yet, as Hegel has shown, no knowledge is possible without it. This indeed in the light of what has already been said above (notwithstanding the defects of the dialectic process in the pursuit to attain it), does not invalidate the argument in this respect. For its denial in the case of the idea of Perfection robs the intellect of all its value and renders reason futile and inept. The idea of an immortal being would be contradicted by that of its mortality if existence be an inseparable attribute of such a being as much so indeed as it is of a perfect one, but only if the two ideas are inseparable. Thus if we put I = an immortal being; i = the idea of such a being; e = that to which the attribute of existence applies. Then the idea of an immortal persistent being that does not exist is non-existent; that is $i\bar{I}\bar{e} = 0$. Also since an immortal being is an idea, $I\bar{\imath} = 0$;

Therefore $$e\{i(I+\bar{I})+\bar{\imath}\bar{I}\}+\bar{e}\bar{I}(i+\bar{\imath}) = 1. \qquad (2.87)$$

But the ontological argument requires that $ei\bar{I} = 0$.
Unless this can be shown to be true, by inference or implication, the argument breaks down. It would be true if $i\bar{I} = 0$, or $e\bar{I} = 0$; that is if the idea of an immortal being is itself an immortal being; or existence is an immortal being, neither of which is by itself true. But if E = the Ego, then what is meant is that $Ei\bar{I} = 0$. This is what the ontological argument actually implies. Namely that the Ego itself being existent as an idea of an immortal being, since existence is a necessary and inseparable attribute of it, is itself an immortal being. An Ego that is non-existent is meaningless. Hence although the idea of an immortal being does not imply the existence of such a being, yet in so far as the Ego is not only such an idea but an actually existing reality containing the idea of an immortal being from which the idea of existence is inseparable,

it must itself be as an existing reality such an immortal being (see p. 215).

The unity of the Ego with the predicate which is the idea of itself as a thinking and reflective being is what renders the argument plausible to those who understand it, even if they do not accept its validity, and convincing to those who do. It follows that

$$EeiI+\bar{E}[e\{i(I+\bar{I})+\bar{\imath}\bar{I}\}+\bar{e}\bar{I}(i+\bar{\imath})] = 1. \qquad (2.88)$$

If it be admitted that the Ego is only an appearance and not a reality so that there is a possible continuity between its existence and its non-existence then the argument does not hold. But if it is real it cannot shade off into unreality without contradicting itself. (*Vide*, however, theory of emergence later, Chapter XV.)

That the Ego which is a self-conscious reality can shade off gradually into nothingness, as the red rays of a radiator when the current is switched off, is what the ontological argument denies. There is an obvious difference between the two cases, that renders the analogy absolutely futile. Although the physiological organism of man can lose its vital energy and disintegrate, as the incandescent body can cease to radiate and grow cold, yet the Ego whose essential attribute is existence cannot have the predicate of non-existence associated with it, for its very essence is thought implying existence, and having the idea of an immortal being with which existence is inseparably connected, it must needs itself be immortal.* Descartes maintained that even in sleep the mind continues active.

The Cartesian principle is that thought implies existence, that is if an idea can exist only in thought, $i\bar{e} = 0$;
Therefore

$$EeiI+\bar{E}[e\{i(I+\bar{I})+\bar{\imath}\bar{I}\}+\bar{e}\bar{I}\bar{\imath}] = 1. \qquad (2.89)$$

If however we add with Berkeley that *esse est percipi*, that is $e\bar{\imath} = 0$; then,

$$EeiI+\bar{E}\{e(I+\bar{I})i+\bar{e}\bar{I}\bar{\imath}\} = 1. \qquad (2.90)$$

It would appear, however, that an idea of an immortal being could exist only in an Ego; and therefore $EeiI+\bar{E}\bar{e}\bar{\imath}\bar{I} = 1$.
Unless there is continuity between the existent and the mortal, that which exists as a reality must always have been, and must for ever be. The premisses are here clearly put forward, and only the idealist would be driven to this last conclusion.

* I do not commit myself to the validity of the argument, but only present it, as so many fail to grasp its true meaning; that the idea of the Ego and existence are inseparably united as *a Verflechtung* or weaving, when the Ego perceives most clearly and distinctly not only that it is, but must for ever be (see passage from Tennyson, p. 54), as its very essence as clearly as the Categorical Imperative in Ethics is undeniable.

It is possible, however, as we have seen above, to arrive at the result relating to the immortality of the Ego without actually accepting either the Berkeleyan or the Cartesian premisses in their entirety, though their acceptance combined renders the result inevitable. (This has often been regarded as the *pons asinorum* of metaphysics.)

It may be said that the argument is only a begging of the whole question, but it is not so; for *the Ego implies both self-consciousness and existence as a unity*. And as other qualities may gradually be dropped these remain inviolably associated with it.

As an analytical judgement it is subject to the apparent vulnerability of Kant's onslaughts upon it. And the assumption that the Ego because known is only phenomenal and not a thing in itself strikes at the root of all knowledge. If one cannot be sure of this then knowledge is impossible. (For further discussion of this, *vide* p. 215.)

§ 2

A similar line of reasoning applies to the idea of freedom. In all three cases Kant falls back on the transcendental ideas. These are of the pure reason. The will, though free to choose between two alternatives, is none the less in the case of a rational being bound by reason, and as such constrained to act in accordance with it. Hence it is determined by it.

If F = absolute freedom; f = rational freedom; r = rational processes; D = determined actions; C = actions constrained by some cause. Then $f\bar{r} = 0$, $r\bar{f} = 0$, i.e. $f = r$; so that rational freedom is a rational process. But $FC = 0$; absolute freedom is incompatible with constraint of any kind; though $r\bar{C} = 0$; any rational process involves some constraint; and $D\bar{C} = 0$; all determined actions are constrained. Hence

$$fr\bar{C}\bar{F}(D+\bar{D})+\bar{f}\bar{r}[F\bar{C}\bar{D}+\bar{F}\{C(D+\bar{D})+\bar{C}\bar{D}\}] = 1. \quad (2.91)$$

Thus a rational act is rationally free though constrained by reason; and though not absolutely free, may or may not be determined. There is no reason why in fact that which is not absolutely free should be determined. Hence rational freedom and reason itself as a rational process may or may not be determined, but must conform with some measure of constraint, without which they would be irrational. This is old ground but newly trodden. In the light of these new symbols it is made more manifest that rational freedom is not necessarily determined. How far the space-time or 'world line' can be subject to the will is considered elsewhere.

Huxley in *Science and Pseudo-Science*, a reply to the Duke of

Argyll in the *Nineteenth Century* made in 1887 (*Science and Christian Tradition*, pp. 120–1), has somewhat confused the issue by begging the question in the following words:

> 'The ultimate form of the problem is this: Have we any reason to believe that a feeling, or state of consciousness, is capable of directly affecting the motion of even the smallest conceivable molecule of matter? Is such a thing even conceivable? If we answer these questions in the negative, it follows that volition may be a sign, but cannot be a cause, of bodily motion. If we answer them in the affirmative, then states of consciousness become undistinguishable from material things; for it is the essential nature of matter to be the vehicle or substratum of mechanical energy.'

This of course is begging the whole question, for there is no reason whatever for supposing that matter, as Huxley understood the term, is the sole vehicle of energy. In fact we know, since Huxley's day, that by the isolation of the electron, Sir J. J. Thomson at Cambridge has shown that matter as conceived in the nineteenth century was not the sole vehicle of energy. Huxley's dogmatic style, inimitable in so many respects, has done much to discredit his sounder views in the eyes of some of his most ardent admirers.

The analysis of this argument fully exhibits the flaw in Huxley's reasoning. Thus if f = a feeling of state of consciousness; μ = things capable of directly affecting the motion of even the smallest conceivable particle of matter (m), where m = material things; V = an act of volition; s = a sign of bodily movement; c = a cause of bodily movement; then it is assumed (1) that an act of volition is a feeling or state of consciousness, i.e. $V\bar{f} = 0$. Also that (2) it is the essential nature of matter to be the vehicle or substratum of mechanical energy; so that virtually $m\bar{\mu} = 0$; i.e. that matter is a thing capable of directly affecting the motion of even the smallest conceivable particle of matter. Yet unless the converse of this latter proposition is also true, that that which is thus capable of moving the smallest particle of matter is itself material, i.e. $\mu\bar{m} = 0$, the conclusion that volition and all states of consciousness become undistinguishable from material things, does not follow, and the argument becomes a *petitio principii*.

Thus, if $f\mu = 0$; then it is stated that $V = v.s\bar{c}$; and $\mu = c$

$$\{f\bar{\mu}+\bar{f}(\mu+\bar{\mu})\}.(Vs\bar{c}+\bar{V}) = 0; \qquad (2.92)$$

or since from (1) $V\bar{f} = 0$; $Vf\bar{\mu}s\bar{c}+\bar{V}(f\bar{\mu}\bar{c}+\bar{f}(c\mu+\bar{v}\bar{\mu}) = 1$, (2.93)

regarding c and μ as identical, which does not affect the argument.

But taking (2) into account we obtain that

$$Vf\bar{\mu}s\bar{c}+\bar{V}.\{m\mu\bar{f}+\bar{m}(f\bar{\mu}+\bar{f}(\mu+\bar{\mu})\} = 1, \qquad (2.94)$$

so that volition though a sign of bodily movement is not material as assumed. If again, $f\bar{\mu} = 0$, then

$$Vf\mu(m+\bar{m})+\bar{V}\{f\mu(m+\bar{m})+\bar{f}(m\mu+\bar{m}(\mu+\bar{\mu}))\} = 1, \quad (2.95)$$

so that volition may or may not be material. Only if $\mu\bar{m} = 0$ is it true that

$$Vf\mu m+\bar{V}.\{f\mu m+\bar{f}(m\mu+\bar{m}\bar{\mu})\} = 1, \quad (2.96)$$

and volition becomes as Huxley supposes undistinguishable from material things.

It is no doubt unnecessary to expand the function fully, except for the purpose of thorough and rigorous demonstration. It being granted that $V\bar{f} = 0$ and $m\bar{\mu} = 0$ it is obvious that if $f\mu = 0$, then $V = Vf\bar{\mu}\bar{m}$; and if on the other hand $f\bar{\mu} = 0$, then $V = Vf\mu(m+\bar{m})$. Huxley's conclusion clearly requires that $\mu\bar{m} = 0$; that is, that things capable of directly affecting the motion of even the smallest conceivable particle of matter, are themselves material. This is what renders his argument a *petitio principii*. The possibility, however, of a feeling or state of consciousness that is not an act of volition whether as a material or an immaterial agent, acting upon matter, is not ruled out of court: nor is the possibility of something capable of directly affecting matter, whether as a material or an immaterial agency endowed with feeling devoid of it, ruled out. Finally the term $\bar{V}\bar{f}\bar{\mu}\bar{m}$ the negation of all these remains possible, as in the Hegelian dialectic, for the effect might after all be transcendental, as Kant maintained.

It has already been shown that a change in the constants g_{mn} in the Einstein equation $ds^2 = \sum g_{11}dx_1^2+\ldots+g_{mn}dx_m dx_n+\ldots$, would be accompanied by an effect equivalent to a change in the field of force. Further what comes to the same thing as a particular case, though from the views I have put forward in the general case, if volition affects the time of a monad, then relative motion ensues, and matter ceases to be the vehicle or substratum of energy mechanical or otherwise.*

§ 3

Evolution.

If we put E = things evolved,
t = things in time,
C_0 = creatures,
β = things transcendental,
then (1) since 'All things that are evolved are in time'

$$E.\bar{t} = 0;$$

* *Vide* also A. A. Robb, *The Absolute Relations of Time and Space*, p. 80.

(2) 'All created things are in time' and 'all things in time are created', i.e. have a first cause, $C_0\bar{t} = 0$; and $t\bar{C}_0 = 0$.

Consequently
$$EtC_0+\bar{E}(tC_0+\bar{C}_0\bar{t}) = 1. \qquad (2.97)$$

Now since according to the Kantian theory 'Time is an *a priori* intuition; it is not a transcendental idea'; then

(3) $$t.\beta = 0.$$

Hence,
$$EtC_0\bar{\beta}+\bar{E}\{tC_0\bar{\beta}+\bar{C}_0\bar{t}(\beta+\bar{\beta})\} = 1. \qquad (2.98)$$

If we express by C_1, The Absolute Being and Creator of all creatures, it follows that if the Deity is transcendental, i.e. $C_1\bar{\beta} = 0$; therefore, He is not evolved. But if He is evolved, as Professor Alexander maintains, then must we deny the Kantian conception of time and admit that it is a transcendental idea. This is also Bergson's contention.

In either case the transcendental is not necessarily the Creator, and there may be some uncreated realities that do not create. For it has not been even implied so far that there is no alternative between the Creator and the creature. Obviously some things may never have been created, but be 'self-existent' entities as the Absolute.

If time is transcendental, then $t.\bar{\beta} = 0$;

Therefore
$$\beta.\{EtC_0+\bar{E}(tC_0+\bar{C}_0\bar{t})\}+\beta\bar{E}\bar{C}_0\bar{t} = 1. \qquad (2.99)$$

Here again even if the Creator be evolved, the Absolute or transcendental may not be so, whether in time or not: unless the Creator Himself is identified with the Absolute. That is if $C_1 = \beta = A$; where A is the Absolute. We obtain, since $C_1\bar{E} = 0$,
$$AC_1\beta EtC_0+\bar{A}\bar{C}_1\bar{\beta}\bar{E}\bar{C}_0\bar{t} = 1. \qquad (3.00)$$

The Absolute is itself the Creator; the transcendental and at the same time created and evolved in time. He would in fact be all that is, except the contradictory of the Absolute which is not in time, and is mere negation and $= 0$. Therefore $AC_1\beta EtC_0 = 1$.

The antinomies in which we thus find ourselves involved are surmounted if time be admitted to be an absolute reality. But as we have seen this cannot now be held in regard to physical time.

There may be absolute time, but it would be transcendental, and entirely beyond our powers of apprehension. Such time is all in which the Creator could be said to evolve, and Alexander's conception of the Deity, unless his position has been entirely misrepresented here, can refer only to such time. To say that the Deity evolves in the time of each one of His creatures is to assign to Him

a purely subjective existence comparable to that of the material world He is supposed to have created; whilst the creature in whom He exists is the true reality; obviously in turn the Creator of the creature's fancy. This in turn would be but a further antinomy little better than any other that we have had before. It is not proposed to criticize Alexander's theory of space and time, but only to point out its relation to the theory herein put forward. (See pp. 93, 186.)

§ 4

It is always possible in the calculus of logic to find values of m and n such that $x^m = y^n$; it is likewise possible that

$$x^m = y^n = z^p = w^q; \tag{3.01}$$

so that in the disjunctive alternatives,

$$x+y+z+w = 1, \tag{3.02}$$

we should have apparently $x = y = z = w$.

Such an arrangement would be like one of those mysterious locks that open only when some particular secret word of the 'Open Sesame' type was found through the choice of the suitable values of m, n, p, q.

Thus the apparent disparities are reconciled in a higher synthesis. It is not generally understood how such a synthesis can be formed, but the logico-mathematical theory of identity affords the solution. At a particular time and at a particular place, or series of such subjective intuitions, the Absolute may be realized. Life and mind as manifestations of the Absolute can emerge as absolute realities from the infinitude of the unknown; and like clouds formed from the azure sky imperceptibly return once more into the infinite chaos of the invisible.

In the 'Emergent Evolution' there is an appearance of something new in time, but not as the resultant of cause and effect, which would be 'Resultant Evolution'. From the latter it would be possible to predict the future if we but knew the laws of the relations of cause and effect in such phenomena. But in the former no such prediction would be possible. There is supposed to be a creative process—which Bergson calls an 'élan vital'—an 'urge', or 'directive activity', that distinguishes it from the purely naturalistic or resultant evolution.

What has been formed by emergence, if it remains, serves somewhat to mould the future by resultant evolution. But a twofold process is at work. The emergence is not the outcome of the past,

though it may serve to mould the future, not of future emergent but of future resultant events. And future emergents though not dependent on it in the sense of being formed by it will depend upon the resultant event or state of affairs for the possibility of its emergence. 'All real synthesis entails new properties which its component factors in their previous isolation did not possess.' 'All real synthesis is creative.'

§ 5

The development of connotation and denotation in time, if not merely a function of time, but of some other factor or factors, as its transcendent nature would imply, produces a distinction between emergence and resultant effects. The resultant being a prescribed function of time. But time, as we have seen, so far as phenomena are concerned is purely relative. The appearance of mind upon which time itself depends for any real signification, must be regarded as transcendent and emergent. Whether it is by chance or purpose the theory of emergence does not reveal; for it is admitted that evolution by selection does take place. Hence the apparent purpose can itself be regarded as the result of successive stages of such selection. That there is a transcendent Purpose can only be established on transcendental grounds. Lloyd Morgan's belief in such a Purpose does not suffice to establish or even to afford evidence other than that of the probability of the truth of innate conviction common with all men of a religious frame of mind. (See p. 186.)

An evolutionary process in time, then, cannot be confined to the particular scale of being in which we happen at the moment to be placed. Such an assumption—howsoever familiar and commonplace—is obviously illusory; and limits without adequate reason the range of evolution, whilst it serves to mystify what clearly results from analogous processes in the infra worlds of the atomic, and the subatomic.

The operations of the chromosomes within the nucleus of the living cell may themselves be the results of evolution. This is not chance though evolutionary development. For the emergence is the result of development in time. But when the actual 'directive force' so evolved is considered, the differences in time between the individual parts of the cell become factors of importance. Thus the mechanism of karyokinesis and subdivision of the nucleus is itself the result of intra-molecular evolution in which the forces set up by the differences of time play an important part. (See Chapter XXIV.) This cannot be explained without an adequate comprehension of

the conditions under which the development of the cell takes place. For in particular the natural selection, such as Darwinism requires, forms a part of the evolutionary process in the cell itself. This cell-mechanism is largely the reciprocal result of the effects it produces in its more developed relations to the fully developed organism. So that by the process of selection in the organism the cell itself is refined. Other factors than natural selection, whether in the organism or the cell, are brought into operation; and the chief of these is the influence of monad on monad arising from the differences of time between them. This is pointed out in Chapter XXIV. The manner in which the inherent qualities are stored up in the nucleus is, I consider, largely dependent upon these differences of time between the individual 'determinants' in the cell. As already shown, these differences may consist of vibrations of time differences resolvable by Fourier's series into harmonic partials. The differences of time between the individual monads, variable as they are, are resolvable into their Fourier components. The possibility of some such effect was given in my book *Origin of Life*, Chapter V, pp. 81–6. This conception is now more apparent in view of more modern considerations; and readily admits of development in the manner indicated in this present work. Just as radiation, including light and all other electro-magnetic phenomena can be accounted for by such changes between monads, so does life itself appear to be the manifestation of differences in time between mônads which if precisely synchronous would never show any signs of life or other striking quality. If monads were all at the same time, beating time as it were, in a pre-established harmony as Leibnitz supposed, they would not react upon or influence each other. But when a difference of phase is admitted between two monads, if the time of each varies around the average time, there will be a force exerted between them. This will be equal and opposite between the two monads. Assuming then, as we have done, that monads are arranged in time in the order in which they appear to be in space: a 'displacement of time', which is equivalent to an acceleration or turn of one relatively to another, will cause a motion of both relatively to each other, and provided the consecutive monads filling space possess inertia, a wave of time-change will be propagated throughout space. (See Chapter XXI.) This displacement or element of time-acceleration takes the place of the old electric displacement in the electro-magnetic theory of light; and the equations of motion should be the same. *A ray of light is thus conceived as 'a time-change' phenomenon that is propagated through space.* What is supposed to be the actual arrange-

ment of monads in space is that particular arrangement relatively to the average or perhaps the Absolute Monad. Each individual monad pictures the change as though it were propagated in its own space-time continuum. The latter is only an imaginary picture of what is actually supposed to occur in the space relations of the Absolute Monad. Without this Absolute Monad we would be led to the most absurd contradictions, such as those with which Lenard has tried to riddle the whole theory of relativity. But by assuming the existence of an Absolute Monad relatively to which all things proceed, absolute space and absolute time regain their former status; though they be merely ideas in one particular monad in relation to which all things must be judged as the final standard of truth.

When the time-acceleration wave is taken up by any individual monad, the tendency, of course, is to alter its time; and a 'pressure' as it were is exerted upon it, whilst if the wave, by whatever mechanism, reaches a sufficiently developed monad the negative of a conscious effort is experienced, namely a perception which is consciousness itself. We have already seen that the reverse process is what takes place when an effort of will is exerted and transmitted to matter in conscious conation. Consciousness such as we are aware of would appear to range within very narrow limits of frequency in such vibrations. What is supposed to be altered here is the subjective time, not the absolute time which is unalterable. But relative time does admit of acceleration, either by gaining or losing upon it: thereby giving rise to all the phenomena of an electrical and magnetic nature with which the physicist has concerned himself, without hitherto considering its precise relationship to other phenomena usually regarded as beyond his ken.

I venture once more to lay bare the relationships* existing be-

* This arbitrary subdivision of the sciences has its advantages no doubt; for not everybody has the time or the desire to study and explore the neighbouring regions of knowledge. Yet most of our difficulties are almost entirely due to these self-imposed restrictions. If a thing is worth doing, it is frequently said to be worth doing well. Nothing can be more obvious or more often emphasized. Nevertheless, it is not sufficiently recognized that if a problem is worth solving, it is worth attacking in the proper manner. And that must mean acquainting oneself first with what is known upon the subject irrespectively of any arbitrary boundaries arising from the limitations imposed by a college curriculum, or the particular restrictions of one's Faith or 'Fach'. If time is necessary for other studies, then time must be found. Other work and engagements must be cancelled. Dinners and meetings should be avoided, and many other instructive entertainments scratched. One cannot study several subjects seriously and spend half of one's time talking or otherwise instructing one's fellows. Somebody must do these things, but obviously one would have to confine oneself in that case to one subject; and not undertake to study other branches of knowledge. I venture to pass these remarks here on account of the cynical attitude that the

tween the various sciences here concerned and to show that life and light vary only in degree, as first indicated in my *Origin of Life* (Chapter XIII) in 1906.

§ 6

A definite conception of the inter-relationship of mind with mind, or monad with monad, is brought out by the idea that monads differ in time and do not exist in space; but may be pictured each in the other's space of consciousness, distant from it only in time. The accelerations in time differences between them serve to explain most of the physical phenomena of Nature. By the analysis into partials through the aid of Fourier's series, all such variations can be made to break up into components that through response afford to each individual monad its own apparent characteristics relatively to others. Each has its own variation function distinct from all the rest, and yet susceptible of responding to partials peculiar to certain groups which serve to make up its own peculiar characteristics. (*Vide* p. 346 et seq.)

Communion of monad with monad through the action of time-acceleration propagated in 'space' is thus the means of communication of monad with monad 'in space'. Yet for aught we know 'space' may form no essential part or condition of the action of mind with mind; and a spaceless communion of soul with soul may still be possible without the agency of 'spatial' intervention. Soul may communicate with soul as though in each other's presence, when the so-called material connexions in 'space' shall have ceased to operate: not by movement perhaps, for movement then would if not in space be void of meaning. Matter would be a closed book to both, as non-spatial existence must needs be, the material of another volume closed to the readers of the book of Nature in material space; though not less real when read with intelligent apprehension of yet a more glorious vision of reality in the ever free transmission of thought in closer or more intimate communion with the Absolute; that Divine spiritual light divested from matter and emancipated from space where the mind soars with freedom, and the spirit of man becomes a part of the Divine.

wide area of my studies might arouse. For example, Aristotle was no myth; but what Aristotle could find time for anybody else with sufficient energy, character, and self-control could do also, provided his inclination permitted him to do so. It is not presumed that the results of the investigations in the two cases would necessarily be comparable. But what has to be insisted upon is—that the task is not an impossible one; that it is a necessary one if the problem is to be properly tackled; and that given the required conditions it is a duty to attack it in this manner. The task may be hopeless, but the effort is worth while.

CHAPTER VIII

THE REAL WORLD OF NOUMENAL IDEAS

§ 1

THE coincidence of a class and the member of a class is essential to the identification of subject and predicate when subject and predicate are of different types. That coincidence may be called a relation or coexistence. It is the problem of the one and the many once more in a disguised form. The one and the many, genus and species overlap, and some types do overlap as in the conscious Ego. Thus the equation (1.55) shows that although there may be classes, whether predicates or not, which are not members of a class, they are not subjects of which anything can be predicated. The same applies to members that are not classes, except that such can neither be subjects nor predicates. Both these cases are extreme, and transcend the power of judgement in experience. As postulates of reason they are perhaps inevitable if beyond experience the one and the many are not one; which is what the pure reason demands. 'The pure concepts of the understanding are those under which all perceptions must be subsumed ere they can serve for judgements of experience in which the synthetical unity of the perceptions is represented as necessary and universally valid.' (Kant *Prolegomena*, Mahaffy and Bernard, p. 62.) The totality of the universe of discourse must include the one that is many in the world of experience and beyond it, as well as the one and the many that do not overlap only beyond experience. The transcendental ideas must be included in the totality.

The totality must include opposites. But the syntheses of the understanding, based upon the categories if carried beyond experience must inevitably lead to contradiction; the antinomies, of which there are far more than Kant actually supposed. We have shown that such contradictions as a Dialectic form a necessary consequence of the law of identity. If a man be looking into a mirror with another mirror behind him, he will obviously see a series of images both of his face and the back of his head, alternately positive and negative. He will in fact see the front and the back view of himself directly positively and negatively, and this *ad infinitum*! What we might call the negative of a proposition would be not its contradictory but something for which the ordinary logic does not provide. The negative of Reality, for instance, is real.

This negative of reality would correspond to the back view, as we shall show.

If $x' = -x$, then $x'.\bar{x} = 0$, since $x.\bar{x} = 0$.

Yet $$x'.(1-x') = -x.(1+x) \qquad (3.03)$$

and is therefore not equal to nothing. That is, the negative does not obey the law of contradiction.

Further $$x.(1-x) = -x'.(1+x') = 0, \text{ or } x' = -1. \qquad (3.04)$$

Hence if x' be the phenomenon, its negative is the noumenon which does not contradict itself. The reality is the negative of phenomena as distinct from the positive appearance which is self-contradictory and continuous with itself. The other side or the *otherness of things* is the reality, we merely see the reverse or wrong side of most things; and that is why we seem to see two contradictory elements in everything, because the negative of what we see is the reality.

If R = the real, and p = phenomena: then if $p = -R$ we get that $p.(1-p) = -R.(1+R)$ and therefore not nothing; i.e. the law of contradiction does not hold. But on the other hand since $R.(1-R) = 0$ we have $-p.(1+p) = 0$. Therefore $p = -1$.

Hence $p^2 = 1$ and $p^3 = p$; or the law of identity holds, although that of contradiction does not necessarily do so for p. Now this is a most remarkable fact, and we need not confine ourselves with the mere negative of the reality, but should reason likewise as regards all the roots of $p^n = p$; or $R^n = R$; or $R.(R^{n-1}-1) = 0$; which can be expressed as

$$R.(R-1).(R^{n-2}+R^{n-3}+\ldots+R^2+R+1) = 0. \qquad (3.05)$$

Now since $R.(R-1) = 0$, the roots of $R^{n-2}+\ldots = 0$ do not enter into the question of reality. But this is not so as regards those of mere phenomena, which are self-contradictory and continuous. The roots of

$$p^{n-2}+p^{n-3}+\ldots+p^2+p+1 = 0 \qquad (3.06)$$

will certainly be of the very greatest importance. Innumerable aspects of phenomenal unreality are possible, according to the roots or order of identity of the phenomenon. We may in fact regard $R^n = p^n$, where n is not greater than 2. The $(n-2)$ values of p never become identical with reality; they are the phenomena properly so called. The phenomenon thus never can be real in the sense of not contradicting itself. This does not mean that it is not some aspect of reality or imaginary, for as we have seen $p.\bar{R} = 0$, if $p = -R$, or other linear function of R; but $R\bar{p} \neq 0$ since $p\bar{p} \neq 0$.

The manifold of an idea, or event, can be represented in an indefinite number of ways. Reality, between which and the phenomenon there is an absolute discontinuity, becomes manifest however inadequately by the various channels opened up through the various grades of identity, in virtue of which continuity becomes possible. So that the Kantian theory of *a priori* intuition, syntheses, as well as judgements which render all experience possible are thus quite explicable as the necessary results of the Laws of Thought. A synthetic proposition, X is Y, becomes possible, because X and Y overlap, as the phenomenal expression of continuity from the X which is Y, to the X that is not Y.

It has to be borne in mind that all those values of a class that may appear to have no real meaning, such as -1, or expressions involving $\sqrt{-1}$, become real or *emerge* in the repeated identity, thus becoming *realized*, in conjunction with other classes of ideas in experience through the intuitions of space and time, and the categories of the understanding, which render experience possible. Experience is then but an 'emergence' of qualities in the multiplication or repetition of identities in time; and in the categories of the understanding; a fusion of class with class, of ideas with ideas so limited or qualified.

The phenomenon which is the negative of reality, or its imaginary counterpart as in the 'space-time continuum' of a fourth dimension—though in Logic in a more generalized form necessarily as the concomitant of thought—implies the Reality which is in fact the transcendental idea of Kant; or if we will the absolute idea or notion of Hegel. Whatever discontinuity then exists between opposites, as that between the real and the unreal, only attests the correlative nature of both in the constitution of thought, the structure of reality from which unreality or the world of phenomena though distinct from it yet necessarily accompanies the real in the totality of things. The two sides, or rather the many sides or facets of ideas, make up the universe of discourse, of truth and apparent truth throughout the realm of thought, and give balance, unity and coherency to the whole. Even the real itself is one-sided and there can be no true comprehension of reality without the recognition of the unreal; or of the unreal without the real.

The real and the unreal together make the totality. For the world of phenomena, of error and illusion, is an essential part of the whole. The grasp which takes in the whole, necessarily includes the sense of incongruity between the opposing elements. A sense of humour perceives the conflicting terms, and seriousness is never adequately such until humour broadens the vision and

illumines the dark or reverse side of all things. One and only one value of X in the equation $X^n - X = 0$ gives the true or real value. But many and innumerable values may be given to it, which although satisfying the equation of identity are nevertheless unreal or untrue. We may if we so chose call these the *humorous solutions*. The latter refer only to the unreal and strictly speaking untrue side of X, the phenomenal. It is by this means that the a *priori* synthesis of a judgement becomes possible in experience, through the categories of the understanding; and that sensuous intuitions of space and time, which we find in us become *a priori* possible.

§ 2

The most interesting application of this principle may be given from the algebraic analysis of Kant's theory of space and time as given in the *Prolegomena*, pp. 212–13 (Mahaffy and Bernard).

If we put s = space; t = time; e = external phenomena; i = internal phenomena; a = apperception; R = Reality: we have (1) The external is in space; and space is external:

$$\bar{s}.e = 0,\ s.\bar{e} = 0,\ s = e.$$

Also (2) Time is internal; and the internal is in time:

$$t.\bar{\imath} = 0,\ \bar{t}.i = 0, \text{ or } t = i.$$

But (3) Space is likewise internal: $s.\bar{\imath} = 0$.

Taking as usual in general

$$(s+\bar{s}).(t+\bar{t}).(e+\bar{e}).(i+\bar{\imath}).(a+\bar{a}).(R+\bar{R}) = 1,$$

it follows that

$$\{it(es+\bar{e}\bar{s})+\bar{\imath}\bar{t}\bar{e}\bar{s}\}.(a+\bar{a}).(R+\bar{R}) = 1. \qquad (3.07)$$

Now in the ordinary course of reasoning we would be inclined to assume that every thing must be either internal or external; in which case every thing would be in time and internal; since $\bar{\imath}.\bar{e} = 0$. This no doubt would be true of every apperception, but not necessarily so of anything else. Hence since time is an apperception, we have (4) $t.\bar{a} = 0$; (5) $a.\bar{\imath}.\bar{e} = 0$.

Therefore

$$ait.(es+\bar{e}\bar{s})+\bar{a}\bar{\imath}\bar{t}\bar{e}\bar{s} = 1, \qquad (3.08)$$

with of course the factor $(R+\bar{R})$.

The point that must be borne in mind here is that so far as the argument goes there is a possibility of things which are neither in space nor time, and therefore neither internal nor external, and accordingly not in apperceptions still existing whether real or unreal. Of course not as phenomena, for all phenomena must be in experience in apperception either in space and time, or time alone.

If we put c = self-contradictory, and p = phenomena, regarding all phenomena as in apperception, and everything in apperception as phenomenal, $p.\bar{a} = 0$, $\bar{p}.a = 0$, i.e. $p = a$. And since all phenomena are appearances or self-contradictory, i.e. $p.\bar{c} = 0$, we have,

$$paitc.(es+\bar{e}\bar{s})+\bar{p}\bar{a}\bar{i}\bar{t}\bar{e}\bar{s}(c+\bar{c}) = 1. \qquad (3.09)$$

Now since reality is non-contradictory, or $R.c = 0$,

$$R.\bar{p}\bar{a}\bar{i}\bar{t}\bar{e}\bar{s}\bar{c}+\bar{R}.\{paitc(es+\bar{e}\bar{s})+\bar{p}\bar{a}\bar{i}\bar{t}\bar{e}\bar{s}(c+\bar{c})\} = 1. \qquad (3.10)$$

If everything is either real, or inconsistent and self-contradictory, the last term goes out. Yet this cannot be affirmed unless consistency is the test of reality.

We see therefore that the familiar notion that everything must be either internal or external would mean that everything is in time and phenomenal, while absolute consistency or reality would be impossible. It must not be supposed from this that Idealism is barred, for the real may be an idea, though not as an apperception in the phenomena conditioned by our sensuous intuitions of space and time. The transcendental ideas are contained in the class R; and I do not think it possible to affirm that Kant could have regarded R as anything but an idea. This we have already discussed (pp. 132, 136). The reality cannot be in either space or time. And if, as we have seen, there can be no continuity between the real and the unreal, an insuperable gulf separates the two. The discontinuity is due to the fact that in the equation $p^n = p$, n is an integer as the law of identity still requires that a class should be identical with itself, and not merely with a fraction of itself, which would be a denial of its logical identity. Only when $n = 2$ is $p.(p-1) = 0$, and the phenomenon can be said to be equivalent to a noumenon as an actual reality (see p. 129). Whatever interpretation the true Kantist will give to this, the doctrine of the transcendental idea 'emerges' directly from the law of identity, and the full logical significance of the Hegelian conception of reality is adequately borne out by it.

When n is greater than 2, identity and continuity become possible so that the thing is unreal. Hence the individual Ego if it is in time and space is not the real Ego; for space (and time more general than space) are both intuitions of the Ego and therefore ideas in it and not it in them.

Fichte maintained that there is but one pure Ego in general, not the individual Ego, but the Absolute Ego. Whether it is essentially

active and creative as he held it to be is another question. The problem of Will and motion as held by Fichte and developed by Schopenhauer is here shown to be a problem in the theory of relativity (see Chap. XIX). But the matter is in space and time; the form is not in space, but still in time; while the Ego which is neither matter nor form, and therefore in neither space nor time, is the reality; the free constructive subject. As the Ego posits first itself, then the non-Ego in distinction or opposition to itself, it then recognizes itself in a higher synthesis as one with the non-ego. This in fact has led up to the Hegelian dialectic, and through it to the Absolute Idea. But although authorities differ as to what Hegel actually meant as to the personality of the Absolute; there can be no doubt as to what Fichte really meant. 'As early as 1797', says Robert Adams, one of the greatest authorities on Fichte, 'he had begun to see that the ultimate basis of his system was the Absolute Ego, in which is no difference of subject and object; in 1800 the "Bestimmung des Menschen" defined this Absolute Ego as the infinite moral will of the universe, God, in whom are all the individual egos, from whom they have sprung . . . God is the absolute Life, the absolute One, who becomes conscious of himself by self-diremption into the individual. The individual ego is only possible as opposed to a non-ego to a world of the senses; thus God, the infinite will, manifests himself in the individual, and the individual has over against him the non-ego or thing' (*Encyc. Brit.*, 11th ed.). 'The individuals', says Fichte, 'do not make part of the being of the one life, but are a pure form of its absolute freedom.' Again, 'The individual is not conscious of himself, but the Life is conscious of itself in individual form and as an individual'. The individual is thus the creation of the Absolute Ego in whose mind all things exist and from whom all beings proceed. The individual minds might be regarded as 'images' of the Divine Mind.* This in fact is the familiar Christian doctrine, in a somewhat vague form. But Fichte meant something far more definite, and indeed quite distinct from the accepted Christian teaching. The creative Ego in us is not the individual ego but the Absolute Ego, pure Ego in general, Absolute Reality, the Divine Spirit, in which we live and move and have our being. 'Thus', says Fichte, 'we reach a final conclusion. Knowledge is not mere knowledge of itself, but of being, and of the one being that truly is, of God. . . . This one possible object of knowledge is never known in its purity, but ever broken into the various forms of knowledge which are and can be shown to be necessary. The demonstration

* The view here put forward is that we can be conscious of the Idea itself or *Λόγος* and not merely images of it.

of the necessity of these forms is philosophy or *Wissenschaftslehre*' (loc. cit.). This teaching inevitably led up to Hegel's doctrine of the Idea, as the Absolute.* But if the individual is a reality, as Hegel insisted, differing in this respect from Fichte, who, as we have seen, regarded the individual as an idea in the Divine Mind, then since selves could not exist as selves within selves there could exist only a multiplicity or community of selves totally distinct from each other.† The unity between them being an 'ideal unity', and not a personality. The pure Ego could not be predicated of the Absolute Idea, notwithstanding Hegel's own protests to the contrary; at any rate according to some of his followers, notably McTaggart. The Oxford school of Bradley and Bosanquet interpreted the ultimate Absolute Idea as an Ego in which subject and object are one. Leibnitz's view that an Infinite Monad unites in relation the whole community or hosts of monads in creation is practically the view held by the rival school.

Our analysis shows that in all matters, including philosophic ones, it is possible to discuss two entirely different things at the same time as though they were the same, and thus arrive at entirely different results with an almost perfect sense of satisfaction at the apparent coherency of each view from its own particular standpoint.

§ 3

The Leibnitzian monadology requires that all things should exist simultaneously in time. But as we have seen this is by no means essential to it. And in the rough and tumble of events, in the long run things tend to settle down to a certain average state; and the time of the individual monad to range round this average time. Whether an absolute time is involved as corresponding to the Infinite Monad, does not affect the Physics of the question. An absolute time, metaphysically speaking, there must be. Yet it is only in relation to the Infinite; and the average time in the universe need not necessarily conform to it. The contradiction is overcome by the fact that time is purely a subjective condition of consciousness, and that in considering absolute time we are still in 'a form of sensuous intuition', and have no right or reason to suppose that even if the Absolute Ego did think in time, as

* Adams points out that Fichte had already prepared the way for the later Hegelian dialectic; and emphasizes how completely the whole philosophy of Schopenhauer is contained in the later writings of Fichte. 'It is not to the credit of historians', he says, 'that Schopenhauer's debt should have been allowed to pass with so little notice.'

† *Vide* chapter on Matter and Mind-Stuff in my book *Origin of Life*.

Bergson maintains in *Creative Evolution*, time would still be a phenomenon and self-contradictory, and not a reality. As we have seen, the fact that it is contemplated by the Absolute Ego would not render it a reality, for the Absolute Ego embraces the totality and therefore the real and the unreal. Time therefore, even if perceived by the Infinite Monad, i.e. the Absolute Ego, still remains unreal, and admits of negation. The unity of thought is in the identity of each of the various monads, as represented by the equation $x.(x^{n-1}-1) = 0$, or $x^n = x$, each root giving an identity, but only one satisfying the condition $x.(1-x) = 0$; as the true expression of reality. There is no unity between the individual monads, even in their relation to the whole, any more than that there is a unity between the roots of the equation of identity. If the Deity has been evolved in time, as Alexander holds, then since time to God is infinite, He can well have attained perfection as has always been held. But this is only one aspect of the Absolute in relation to His Personality, which presumably must be in time if it means anything. Here, of course, we encounter the familiar controversies as of old, whether 'Filioque' or Arian matters not. It is going beyond the limits of the present work. The personality of the Infinite is not to be identified with the Infinite Personality of the Deity; for the Infinite can obviously have no personality, as it would have to include all other personalities or monads, selves within selves, a contradiction which would render it an unreality. The Infinite Personality does not contradict this, but is quite in accordance with it. The unity is in the idea connecting the Infinite monad with the individual ones. Is there no subordination of God to the idea implied thereby, as Professor Clement C. J. Webb apparently asserts there should be? (*vide* 'God and the World', *Journal of Philosophical Studies*, July 1927). God is not everything and everywhere, nor is the World God as Spinoza required, for He would be a contradiction and unreal. Nor is He necessarily the whole of reality. If God is the Absolute, but the Absolute is not necessarily God, then we are met with the same problem of the one and the many; the link between them is in the relation $x^n = x$. The relation is absolute, but only because $x.(1-x) = 0$ gives the Absolute unity in which the consciousness of subject and object become identical as one.

The 'universal now' is the time of what might be well called the Universal Monad. It is approximated to by the average time of the Universe, acquired through the settling down of monads throughout infinite time. Yet this time, like all time relative to it, is only 'a form of sensuous intuition' in which all monads think; and is no more

real than space. The space of the Universal Monad is likewise the so-called 'absolute space' that in truth is also a 'form of sensuous intuition'; no more real and far less general than time itself. The Kantian doctrine of space and time was supposed to require that the noumena should never be representable to cognition on that account. But as we have seen this is assuming more than we are justified in doing; for it is conceivable that the noumenon should become known as a phenomenon, not of course as a contradictory thing in the sensuous intuition of space or time, but in some transcendental sense. Kant confines phenomena to the senses conditioned by space and time, and therefore infers that the noumenon never could be known as a phenomenon. He then concludes that the noumenon never could be known at all! This as we have seen is quite fallacious. We have no right to suppose that cognition is possible only through the five senses with which we are here endowed. A person born blind might well imagine that sound and touch covered the entire field of possible channels of experience. And so with those who lacked any other sense or faculty. The transcendental power of thought attainable by analytic calculus is almost unimaginable to the unlearned, as reading and writing may be to the illiterate. This, no doubt, is what Sir Oliver Lodge insists upon. The typewriter no more does the thinking than the brain can be said to do so; it is the mind behind the brain; the transcendental noumenon, as distinct from the phenomenon; when only the mind is its own phenomenon as self-conscious of itself. Not brains but the Ego is the guiding, ruling, responsible and free agent which can distinguish right from wrong. The brains of a fox-terrier will never intuit the problem of creation. And the brains of the poor invertebrate fly might equal in skill of evasion those of the most expert cricketer or footballer. No 'varsity' blue, much as his nimbleness is to be admired and encouraged, has ever excelled in excellence and prowess the achievements of the humble but malicious flea. And the subtlety and mental alertness of the philosopher and mathematician must mean something more than mechanical cog-wheel gearing and manipulation. There is the transcendental Ego of the Spirit, with its abode in the higher plane of the realm of pure thought, a transcendental idea, not the servant of matter, but the master of matter and of himself, of that envelope material or spiritual which is the link between him and the rest of things.

In this regard the theory I have put forward of the noumenon as the negative of the phenomenon, or the correlative of the remaining

negative and imaginary roots of the logical equation of identity, is of particular interest as applied to the conception of the Ego.

Thus if E = the Ego as a noumenon in Reality, and E' = the phenomenal Ego, we have in general $E^n = E$, or $E.(E^{n-1}-1) = 0$ (3.11); and for $E = 1$; $E.(E-1) = 0$. If n is an odd number, as it would be when the self perceives itself both as subject and object, and E is something distinct from these, perceiving at once the relationship of both. Here $E = -1$ is a possible solution of the equation of identity.

If $E' = -1$, i.e. $E = -E'$, $E'.(E'-1)$ is not $= 0$ (3.12). In other words, the phenomenal Ego is and is not itself, and a contradiction; but the real Ego is no contradiction but itself and nothing else. There is no reason to suppose that this absolute Ego is the only Ego. This would be sheer solipsism. The existence of other minds is as certain to us through induction as the instinctive ideas of our own consciousness. From inductive analogy of a very high order of probability it follows that, although when A contemplates B, B is merely an idea in A's mind, yet B's behaviour must convince A that they are two different persons, and that there is likewise a personality behind A's idea of B that is totally distinct from A's own. The coherency renders the probability of their distinctness a moral certainty. But this only in the phenomenal world. It cannot be said that in the transcendental world they are not one, as the transcendental ideas of truth, of goodness, of beauty, and of love are one; and indeed of all 'pure universal ideas'. Fichte, as we have just seen, held them to be one. Not as a solipsist, for his own individuality was not that of the absolute Ego, but more than an expression or image thereof; so that we are all not merely 'reflexions' as it were of the Divine, but ideas in the Divine Mind itself. This indeed in substance was the theory that I ventured to put forward in the *Origin of Life* (1906). My subsequent studies have convinced me that innumerable Egos, as Leibnitz supposed in the 'Monadology', must form the basis and the true structure of reality. Each monad is a reality in itself; and although the Kantian principles at first sight do seem to lead to Fichte's views, there can be no doubt that inductive reasoning leads us to suppose a difference even in the transcendental Egos. Freedom could have no meaning, nor could the Ego that was other than itself in its ultimate reality be regarded as a reality, which it undoubtedly is, if this were not so.

Although the equation under consideration does not enable us to decide this question with any measure of rigorous reasoning other than induction, so that the view put forward is a mere hypo-

thesis, yet other reasons render the distinctiveness of personality tolerably certain. If certainty is possible and the Ego is not merely a phenomenon, but also a noumenon contemplating the truth, and in fact truth itself as a reality, the distinctiveness of the individual ego as a thing-in-itself, or noumenon, cannot be doubted.

Finally the ethical argument employed by Kant and Fichte—by one as the ultimate basis of Theism, by the other as the basis of the belief in the multiplicity of individuals—makes one feel perhaps the overwhelming weight of the categorical imperative as a law of the physical and spiritual as well as the moral order, lightening if it does not quite remove 'the heavy and the weary weight of all this unintelligible world', whilst giving ballast and equilibrium to the whole as we return to a more familiar aspect of things.

§ 4

The Problem of Evil.

The inevitable antinomies to which the understanding is directly led in dealing with the problem of evil are avoided by having recourse to the transcendental, which is the proper and truly logical solution of the difficulty. It is held that in so far as evil does undoubtedly exist, and that such evil is incompatible with the idea of an absolutely good Creator of the world, an absolutely good Creator could not have created it. This argument presupposes that an absolute alternative exists between the ideas of good and evil, an alternative that is by no means essential. To an Infinite Being the attributes of good and evil may not apply. Such a being could transcend both good and evil, since these terms might be applicable only to finite creatures. The Creator need not be good or bad in the sense in which these terms are applied to ordinary mortals, and infinite wisdom, like infinite justice, may imply much that to finite intellects may seem unjust and bad.

Thus if g = absolute good, g_1 = partial good, and e = absolute evil and e_1 = partial evil.

(1) Absolute good and absolute evil are incompatible, i.e.

$$g.e = 0,$$

therefore

$$g\bar{e}+\bar{g}e+\bar{g}\bar{e} = 1. \qquad (3.13)$$

Also (2) partial good is partial evil, and vice versa partial evil, partial good. $g_1\bar{e}_1 = 0$, $\bar{g}_1e_1 = 0$, or $g_1 = e_1$ $\therefore$ $g_1e_1+\bar{g}_1\bar{e}_1 = 1$, so that every thing is either partial good and partial evil, or neither one nor the other.

Now (3) Absolute good is not partial good; and absolute evil is

not partial evil, i.e. $gg_1 = 0$, and $ee_1 = 0$, so that if we combine (1) and (2) with (3), we obtain that

$$\therefore\ g\bar{e}\bar{g}_1\bar{e}_1+\bar{g}[e\bar{g}_1\bar{e}_1+\bar{e}(g_1e_1+\bar{g}_1\bar{e}_1)] = 1. \tag{3.14}$$

(4)

Therefore every thing or being is either absolutely good or absolutely evil, or both partially good and partially evil, or else it is none of these.

Before the argument can be considered to apply to an Infinite Being and First Cause, it would be necessary to bear in mind the factors of omniscience and perfection; as $(O+\bar{O})\ (P+\bar{P})$, where O and P stand for omniscience and perfection. The favourite argument of atheists thus exhibits its weakness.

Since $O\bar{P} = 0$, and $\bar{O}P = 0$, i.e. an omniscient being must be perfect, and a Perfect Being omniscient, it follows that

$$OP+\bar{O}\bar{P} = 1. \tag{3.15}$$

(5)

If we denote the cause of partial good and evil by C_{g_1}, then since such a cause cannot be absolutely good if omniscient, so far as these terms apply to finite beings,

$$C_{g_1}.g = 0;\ C_{g_1}e = 0; \tag{3.16}$$

(6)

and since a Perfect and Omniscient Creator would not be the cause of partial good such as exists, then

$$OP.C_{g_1} = 0. \tag{3.17}$$

(7)

Hence from (4), (5), it follows that

$$C_{g_1} = v.\bar{O}\bar{P}\bar{g}\bar{e}(g_1e_1+\bar{g}_1\bar{e}_1). \tag{3.18}$$

The cause of partial evil is not absolute good, nor absolute evil; and it is neither omniscient nor perfect; and it may not be partially good or partially evil. Not being omniscient and perfect, evil might be something, owing to its imperfection in this respect, over which it might have no control; or so far as the argument goes, the cause of partial good and evil might be evil that not being omniscient could not prevent good from evolving as in the fierce struggle for existence in Nature. And again though not omniscient and perfect it might yet transcend all evil and all good. Moreover, absolute good and absolute evil may have nothing whatever to do with the cause of partial good and evil. But if it be held that absolute good or absolute evil cannot be the cause of partial good and evil that accompanies it, i.e. $gC_{g_1} = 0$; $eC_{g_1} = 0$, and further, that the cause of partial good and evil may well be regarded as partially one and partially the other; the ultimate cause of partial good and evil must be either itself partially

good and partially evil, and neither omniscient nor perfect; or else beyond good and evil, in which alone is its perfection and omniscience to be found.*

It must be honestly admitted, however, that (6) is going beyond experience, since we cannot rightly say that an omniscient and perfect Being would not think it right, though He could foresee the result, to create in the interests of good that which would involve partial evil. For instance, the only path to the goal of perfect good might be through partial evil, from the nature of reality, in which Perfection itself as the best possible can be realized only in the gradual elimination of evil in the attainment of good.

This is the view that S. Alexander has evolved of the Deity as a Being in process of development towards Perfection. Absolute goodness could then be predicated of such a being, but not omniscience and perfection.

The alternative then is between such a being and the transcendent, omniscient and perfect being conceived by Kant beyond good and evil, of whom good and evil cannot in any degree be predicated; and the All-Wise, All-Good, All-Perfect and Omniscient Being, the God of Plato, though not perhaps of Abraham, Isaac and Jacob, but of most thinking men of to-day, whose Divine Wisdom none can explain but whose Real presence few will deny. To say that Infinite Wisdom and Justice cannot do what human intelligence cannot comprehend as either just or wise in its limited capacity, is obviously to push human faculty beyond its limits. The transcendent element is in the Deity being beyond good and evil, or else in good and evil not being incompatible with transcendental Wisdom, Justice, and Goodness; and though these may seem to be truly incompatible, they may in turn be transcended to avoid inevitable inconsistencies of the understanding as narrowed by experience.

* If we put $x = g$, $y = e$, $z = g_1$, $w = e_1$, $p = OP$, $q = Cg_1$: then the premisses are (1) $xy = 0$, (2) $zx = 0$, (3) $yw = 0$, (4) $pq = 0$, (5) $qx = 0$, (6) $qy = 0$, and (7) $z = w$.

Then $x\bar{y}\bar{z}\bar{w}(p+\bar{p})\bar{q}+\bar{x}y\bar{z}\bar{w}(p+\bar{p})\bar{q}+\bar{x}\bar{y}(zw+\bar{z}\bar{w})(p\bar{q}+\bar{p}q+\bar{p}\bar{q}) = 1$ (3.19)

is the complete solution. Therefore p is not incompatible with y, that is, OP with e; or omniscience and perfection with absolute evil; unless $p\bar{x} = 0$ or $p\bar{z} = 0$, or $p\bar{q} = 0$, that is unless omniscience and perfection imply absolute or partial good, or else can be the cause of partial good and evil. This would I think readily be granted.

The cause of partial good and evil, i.e. q is neither absolute good nor absolute evil; nor is it necessarily partially good or partially evil, but possibly none of these, and transcendental, unless $\bar{x}\bar{y}\bar{z}\bar{w} = 0$: which it need not be, since it may not be good or evil at all.

CHAPTER IX

THE CRITERION OF CERTAINTY

KANT tries to riddle the ontological argument, and idealism generally, with objections by pointing out that the ontological argument is based upon an analytical proposition; and that therefore it is impossible to infer from it the existence of a Perfect Being, even if we had the idea, as it would require a synthesis beyond experience.

If P = a Perfect Being, e = things that exist, then $P.\bar{e}.P = 0$; that is 'A Perfect Being that does not exist, if Perfect cannot exist'; in other words, as we always express such a proposition, 'No Perfect Being that does not exist is Perfect'. That is a 'Perfect Being cannot be a thing that does not exist'. In other words, *if it is anything* then 'A Perfect Being is a thing that exists'. These are, of course, based upon analytical notions of the nature of a Perfect Being. Yet if we deny the truth of this proposition the denial becomes nonsensical. If a being is not perfect it may or may not exist. If it is perfect it must exist, if our idea of perfection has *any meaning*. Hence God, who by definition or description is a Perfect Being, must necessarily exist, since to deny His existence, or even to admit the possibility of His non-existence, is to deny His Perfection; that is the predicate of non-existence would be applied to something else, and not to God Himself.

Consequently it might appear that some analytical judgements are undeniable, notwithstanding Kant's assertion to the contrary. It has been held to be as absurd to deny the truth of Descartes' 'Cogito ergo sum' as to admit the possibility that a Perfect Being might not exist. For if reason has any value it must attest to the reality beyond experience as a transcendental idea. Noumena and phenomena in some cases become one, else knowledge and truth become impossible. Kant's fictitious distinction between analytical and synthetic knowledge, and the further gap which he supposes to lie between all noumena and phenomena, would seem to render reason inopportune, ineffective, and inept. Only if perfection has no meaning could existence be denied to it. Nevertheless the fact that we cannot deny that a Perfect Being, if there be such, must exist, does not prove that He exists; it only proves that *unless our reason is at fault*, He must exist. Unless God is a *mere name*, to say that he does not exist is an Irish bull.

If $x \equiv P.\bar{e}$; then since $x.P = 0$,

$$\bar{x}.\{P.e+\bar{P}.(e+\bar{e})\} = 1. \qquad (3.20)$$

Therefore $x = 0$; or $P.\bar{e} = 0$; $P = ve$, i.e. a Perfect Being must exist. The point is that in logical terms the class 'Perfect Being' necessarily includes or is included in the class of existing things. Whatever 'Perfection' might really mean, existence is a quality that cannot be denied to it. Descartes held that *the fact that I have the idea* renders doubt on the subject impossible.

If p = anything that is possible, then since 'Everything that exists is possible', $e.\bar{p} = 0$. But 'if a Perfect Being is possible He exists', $P.p.\bar{e} = 0$.

Hence $$P.(p.e+\bar{p}.\bar{e})+\bar{P}.(p.e+p.\bar{e}+\bar{p}.\bar{e}) = 1. \qquad (3.21)$$

Now we have the possibility of a Perfect Being not being possible and therefore not existing. But what kind of perfection would that be that was impossible? Obviously perfection must be confined within the bounds of possibility; and the latter would appear to be as necessary an attribute of perfection as existence itself.

Since all that exists is possible, and some things do exist, some things at least are possible. Now if by a 'Perfect Being' we mean 'The Best of all possible beings', and since by 'best' we mean a being that is not only possible but also not merely imaginary, but real, we associate both possibility and existence with perfection at the same time. It is therefore unnecessary to say that a Perfect Being must be possible in order that He should exist. A Perfect Being must at once be both possible and existent, else it would be only imaginary and not perfect.

There cannot be imperfect things that exist and yet perfect things that do not, unless nothing is perfect, which is so far possible. They would obviously be in the wrong classes. The class of non-existing things obviously cannot then contain a Perfect Being; but the question at issue is whether the class of existing things must necessarily do so; and the ontological argument implies that it does, unless reason is at fault and certainty in any matter is unattainable.

If a Perfect Being is only imaginary $= P_0$, such a Being could not be perfect. Therefore

$$P_0P = 0. \qquad (3.22)$$

But $P_0e = 0$, that is, an imaginary Perfect Being does not exist in the sense a Perfect Being would be said to do so.

In ordinary circumstances $x = vy$ or $x\bar{y} = 0$, i.e. if x then y is conditional upon there being an x. Similarly it might be held that $P\bar{e} = 0$ would mean that if there be a Perfect Being then He must exist; but that there might be no such Being, the term 'Perfect Being' being only a name. It also involves a description and

existence is one of the attributes of a Perfect Being. If I imagine such a Being as possible, my reason tells me that he must already exist. Just as if I imagine that I see the sun with my eyes shut; but when I open them and the light beams in my sight I know that it is there. It is then perceived and no longer imagined. It is a phenomenon whatever the reality behind it might be. But Infinite-Perfection necessarily implies existence without which it would be neither infinite nor perfect. It will not do to say, that though $P\bar{e} = 0$ it might be because $P = 0$.

In $x\bar{y} = 0$, $x = 0$ is a possible solution; but in $P\bar{e} = 0$, $P = 0$ is not so. The implication is that in dealing with rational ideas we are dealing with reality: otherwise the argument would break down.

St. Thomas held that we could never from the idea pass to the reality, and Kant likewise maintained that we could not reach the noumenon from the phenomenon. But the Hegelian philosophy, as we have shown, requires that the idea and the reality should be ultimately one; else truth would be chimerical.

With all deference also to Whitehead and Russell, some types do and must overlap, as the Cantor theory of infinity requires. Again if $P = f(e) = f(1)e + f(0)\bar{e}$, so long therefore as $f(1)e \neq 0$ $P = f(e)$ is a propositional function; then $P \neq 0$.* (3.23)

Of course, in general the definition of a class does not justify the inference that the class has members, as Mr. R. F. A. Hoernlé has well insisted (see his paper *Proc. Aristotelian Soc.*, 27 Nov. 1922, 'Notes on the Treatment of "Existence" in Present Philosophical Literature'). But if the definition implies the existence of at least one member as a necessity, the definition, if adequate, does imply the existence of this one as a member. That 'than which nothing greater can be thought' must be found in the class of existent and not in the class of non-existent beings. The jump from necessary thought to reality is possible only if certainty in reason is admitted. This has already been discussed elsewhere.

The distinction between a descriptive function $y = a \log x$ and the propositional or truth function in which x has a particular value is of very great importance. An equation is a propositional function, since it is true for only certain values of x. The descriptive function is a relation which holds for any value of x.

The function $s = f(t)$ if true for any value of t is a 'descriptive function', but if true for only particular values of t it is a 'pro-

* Or if $P = f(e_1e_2e_3 \ldots e_n)$ when $e_1, e_2, e_3, \ldots, e_n$, represent all possible modes of existence, then

$$P = ve_m, \text{ when } ve_m = 1. \quad \therefore P \neq 0. \tag{3.24}$$

To get at the root of things we must ascertain the basis of certainty: That which satisfies a truth or propositional function is a fact and exists.

positional function' and when these values of t are inserted it becomes a 'truth function'.

According to Russell, though he seems to entertain some doubts upon the subject (see *Introduction to Mathematical Philosophy*, p. 203), the ontological proof of the existence of God is fallacious. He says it 'fails to realize that existence can only be asserted of something described', not of something named, so that it is meaningless to argue from 'this is the so-and-so' and 'the so-and-so exists' to 'this exists'. But a Perfect Being is not merely something named but something described to which existence is a necessary attribute, for there would be no perfection without existence, i.e. $\bar{e}P = 0$. Therefore

$$P\bar{e} = 0 \qquad P(1-e) = 0$$

or
$$e = \frac{P}{P} = \frac{1}{1}P + \frac{0}{0}\bar{P}, \tag{3.25}$$

$P = f(e)$ is a propositional function.

That is an imperfect being *may* exist; but a Perfect Being *must*. *If Perfection has meaning* $P\bar{e} = 0$ becomes a propositional function, P as a Perfect Being is not merely a noun but a description. Mr. Joad says that existence cannot be predicated of God, presumably because He is Infinite; but this I do not agree with, for existence must be an attribute of perfection, the highest form of existence and perfection being identical.*

The human mind can thus appear to acquire a knowledge of the truth; the Absolute Truth, noumenon and phenomenon to become blended into one. Some analytic judgements are of such a nature that self-evidence attests the truth without any synthesis beyond experience. The post-Kantian move towards Idealism shows a profound dissatisfaction with the critical philosophy in this respect. Fichte, Schelling, and Hegel marked a decided deviation from the path which Kant had indicated.

The critical philosophy postulates much that it would gladly dispense with. There is no getting over the fact that what Kant calls the Pure Reason is inevitably bound up with phenomena on the one hand and noumena on the other.

The criterion of certainty does not seem to be properly discussed in the so-called critical philosophy. For attempts to discredit Descartes and Leibnitz only serve to show that Kant himself was labouring under the constraints he had himself imposed, and made

* 'It does not follow from the definition of a class', says Mr. Hoernlé, 'that it must have members' unless the definition implies at least one member; and that in so doing it has meaning. This is also the view taken by Professor A. E. Taylor, the greatest modern exponent of Theism.

use in his own reasoning of those very principles that he denied. He imagined something beyond experience, such as the non-phenomenal noumenon, the necessity of which he could not prove, and the absence of which he could not assert; but the possibility that a thing-in-itself could be perceived is what he did deny, and that only because he chose to set up by definition something that was beyond experience, and would ever be beyond it; notwithstanding his own claims to a knowledge of the truth through the 'transcendental ideas'. This obviously does not, however, apply to the idea of a Perfect Being which from its very nature is clearly an exception.

The point is that if I am certain and yet know from experience that however certain I may once have been, let us say, of the truth of the proposition that the sum of any two sides of a triangle is greater than the third side (in a plane triangle), there is no guarantee of the truth of anything I may now feel certain of, for I know that the proposition is not strictly speaking true, yet there are some propositions the truth of which cannot be disputed or in any way doubted without renouncing all claims to a basis of rational discourse of any kind. Such for instance as 'Cogito ergo sum', or that 'Existence is a necessary and inseparable attribute of perfection in a Perfect Being'. The fact that I might be deceived in some things of which I once was quite certain does not entitle me to doubt even these two propositions, or either of them separately, for to doubt the former at least would be, as Descartes said, to doubt that we were doubting—a procedure that is excluded by a kind of estoppel of reason itself—whilst the latter is likewise a self-contradiction in terms.* There are obviously some propositions the truth of which I must claim to be certain, about which I cannot admit the possibility of my being deceived, quite apart from all questions of a moral as distinct from an absolute certainty forming the basis of all reasoning.†

If G = the proposition 'I believe that God exists',

d = 'I admit the possibility of my being deceived about something',

c = 'I am certain of something',

Then in general $$(G+\bar{G}).(d+\bar{d}).(c+\bar{c}) = 1. \qquad (3.26)$$

Now if I do not believe that He exists, I must admit the possibility of my being deceived, since only the belief in God,

* These are really Russell contradictions.

† The following further examples are given as examples of close but difficult reasoning when the law of contradiction is assumed to hold good.

the Perfect Being, could give me the assurance that the idea of Perfection alone entails. But whether I believe in God or not, my certainty of something is quite incompatible with the admission of the possibility of my being deceived about it. Yet one has no guarantee without the other.

Therefore we have

$$(1)\ \bar{G}.\bar{d}.c = 0;\ (2)\ c.d = 0. \tag{3.27}$$

and accordingly

$$G.(d.\bar{c}+\bar{d}.c+\bar{d}.\bar{c})+\bar{G}.(\bar{c}.d+\bar{c}.\bar{d}) = 1; \tag{3.28}$$

otherwise $$c.\bar{d}.G+\bar{c}.(d+\bar{d}).(G+\bar{G}) = 1.$$

Hence so long as I am uncertain I must admit the possibility of my being deceived, whether I believe or do not believe in God. But if I am certain of something, then I cannot admit the possibility of my being deceived about it, and must believe that God exists.

Hence although error is possible even if I believe that God exists, yet it is possible to have absolute certainty when and only when we believe in God, for only through the contemplation of the idea of Perfection is this certainty possible. Descartes here gets at the bedrock of existence and the possibility of knowledge.

I cannot refrain from asserting that Kant does not do justice to the Cartesian argument; nay more, he gives us no criterion of his own that would serve to take its place, except his transcendental ideas of the Pure Reason.

The knowledge of the truth is a knowledge of that which is at once noumenon and phenomenon. Whether it is an intuition or not has nothing to do with the question.

The principal proposition underlying the whole argument is that, 'if there is no such thing as Perfection, then no guarantee for certainty exists; for only in the idea of Perfection is certainty possible'. In other words, using the symbols G, P and c in the same sense as before, $$\bar{G}.P.c = 0. \tag{3.29}$$

Therefore

$$c.P.G+c.\bar{P}.(G+\bar{G})+\bar{c}.(G+\bar{G}).(P+\bar{P}) = 1. \tag{3.30}$$

We may therefore combine the two equations just obtained, and arrive at the result that

$$c\bar{d}G(P+\bar{P})+\bar{c}(G+\bar{G})(d+\bar{d})(P+\bar{P}) = 1, \tag{3.31}$$

or, $$c.\bar{d}.P.G+\bar{c}(P+\bar{P})(d+\bar{d}).(G+\bar{G}) = 1;$$

since $$c\bar{d}\bar{P} = 0.$$

We may of course simplify the equations by putting $\bar{d}.\bar{c} = 0$;

since if I do not admit the possibility of my being deceived I must be certain. Consequently we may write the above equation

$$[Pc\bar{d}G+(P+\bar{P})\bar{c}d(G+\bar{G})] = 1. \qquad (3.32)$$

Hence it is possible that there is no God; but in that case I could not be certain of anything, not even of this possibility. It is likewise possible that there might be a God; and in that case I might or might not be deceived in my belief about Him. But if I am certain of anything according to the terms already defined, then I must believe in God, for the reasons already given that no certainty is possible without the idea of Perfection and the realization of the Idea. Though deception is still possible if I am not certain of a thing, yet I am certain that I am not deceived when I am certain.

It is noteworthy that we need not assert that if I am certain of anything whatever, I cannot admit the possibility of my being deceived. In other words $c.d = 0$ is not necessarily true. We may confine ourselves to the statements (1) $c.e.d = 0$; that is, the certainty is confined to Descartes' 'Cogito ergo sum', and (2) $c.\bar{e} = 0$ when e stands for 'that I exist'. Therefore $c.d = 0$, only because of (2), so that

$$ce\bar{d}+\bar{c}(d+\bar{d})(e+\bar{e}) = 1. \qquad (3.33)$$

Hence, since in general,

$$(G+\bar{G})(d+\bar{d})(c+\bar{c})(e+\bar{e}) = 1;$$

if we admit the validity of the ontological argument

$$(3) \quad \bar{G}\bar{d}ce = 0,$$

we have

$$G\bar{d}ce+\bar{c}(e+\bar{e})(d+\bar{d})(G+\bar{G}) = 1. \qquad (3.34)$$

Hence certainty, as before, depends upon the belief in a Perfect Being.

If you doubt the existence of God you do not on that account doubt your own existence, but you are deprived of the grounds of all certainty, even of your own existence; for only the belief in a Perfect Being could give you the assurance that the idea of Perfection alone entails.

This argument may appear to the uninitiated to have all the appearance of an ingenious sophistry. The Agnostic might even admit the validity of the whole argument and yet conclude by admitting that he was not certain of anything, not even of his own existence. In that case he would be justified, no doubt, in saying that there might or might not be a God; that he might or might

not be deceived; that he might or might not really exist, whatever that might mean; but he would have to abandon his idea of truth, and the idea of his capacity to grapple with it through reason, for he is inevitably driven to Herbert Spencer's fathomless abyss of the unknowable, in which reality has no place, and reason is the mere phantom of the imagination. In such a case indeed ignorance is bliss and it would be folly to be wise; for we then know not anything, not even that we know this.

The fundamental premiss here of course is $\bar{G}_1\bar{d} = 0$; G_1 now stands for 'if God exists', that is, that if there is no God then at least I may be deceived, i.e.

$$G_1d+G_1\bar{d}+\bar{G}_1d = 1, \tag{3·35}$$

so that it is admitted that even if there be a God I may or may not be deceived. This is quite different from Descartes' assumption, that if there be a God He would not let me be deceived,

i.e. $$G_1d = 0;$$

or $$G_1\bar{d}+\bar{G}_1d+\bar{G}_1\bar{d} = 1. \tag{3·36}$$

From which it follows that if I am deceived then there cannot be a God, for if there were He would not let me be deceived. Whereas in the previous case, if I admitted the possibility of my being deceived I might still believe or not believe that God exists; but in the latter case even if I had grounds for certainty I would not have to believe in God, unless the belief in God and the belief in the possibility of my being deceived are absolute alternatives, since the 'third alternative' holds that if there is no God I might still not be deceived.

If, however, in Descartes' case $$\bar{G}_1\bar{d} = 0,$$

then $$G_1\bar{d}+\bar{G}_1d = 1. \tag{3·37}$$

Here there would be no alternative between the possibility of my being deceived and my believing in God, so that if I believe in God I cannot doubt my certainty; and if I did I would have to admit that I did not believe in God. This reduces the issue to Newman's dilemma between Theism and unbelief.*

The beauty of this method is in the perfect candour with which the whole case is stated; and the cards, as it were, are laid on the table. If there is any difference of opinion, and there often must be,

* *The Grammar of Assent*, by John Henry Newman. He maintains that there is no alternative between Theism and Unbelief, and that the mind goes from a high degree of moral certainty or probability to absolute certainty that compels assent, through the process of what he calls the *illative sense*.

the points are clearly set out and traced to their fundamental source.

In Newman's statement of the case there is no alternative between theism and atheism. But in the previous arguments agnosticism was not ruled out of court. Only to him that is certain is the belief in God imperative, and he that believes not has no grounds for certainty of any kind whatever. Knowledge is possible only with belief, and that once more without belief ignorance as we say is the only source of happiness.

§ 2

The ontological argument may be pursued differently, thus:

If p_1 = possible things,

$p_2 = e$ = things that exist,

p_3 = necessary things,

then since

(1) Everything that is necessary is possible, i.e. $p_3.\bar{p}_1 = 0$;

also (2) Everything that is necessary exists, i.e. $p_3.\bar{e} = 0$;

and (3) That which exists is possible, i.e. $e.\bar{p}_1 = 0$.

Consequently we obtain that

$$e.p_1.(p_3+\bar{p}_3)+\bar{e}.(p_1+\bar{p}_1).\bar{p}_3 = 1. \qquad (3.38)$$

Now if God is possible He must exist, i.e. $G.p_1.\bar{e} = 0$.

Therefore

$$G.\{p_1.e(p_3+\bar{p}_3)+\bar{p}_1.\bar{e}.\bar{p}_3\}+\bar{G}.\{p_1e(p_3+\bar{p}_3)+e\bar{p}_3(p_1+\bar{p}_1)\} = 1. \qquad (3.39)$$

Hence although it be true that if God is possible He must exist, it would not follow from the premisses so far put forward that He must exist as a necessary being; for, even if possible and therefore existent, it would not follow that He was also a necessary being; for this possibility might be contingent. Moreover, a necessary being, although not only possible but actually existent, would not necessarily be God. This is the rigorous logic of the question.

Instead of trying to prove the necessity from the possibility, the proper procedure is to infer the possibility from the necessity of His Nature, for a necessary being must at once be possible and existent; and His necessity is, as we have seen, an essential attribute of his being $G\bar{p}_3 = 0$. Therefore we obtain,

$$G.p_1.ep_3+\bar{G}.\{p_1e(p_3+\bar{p}_3)+\bar{p}_3\bar{e}(p_1+\bar{p}_1)\} = 1. \qquad (3.40)$$

A necessary being is, so far as the ontological argument goes, not necessarily God. Other considerations render it imperative that an uncaused being could be none other than God Himself, but the

ontological argument as regards a necessary being gives us no clue whatever to it.

(1) If now we put

C = feeling of certainty,
T = that which implies truth,
P = a purposive intelligence,
K = chance,

then, since certainty implies truth, (1) $C.\bar{T} = 0$.
Also, if truth and chance are incompatible, (2) $T.K = 0$.
Moreover, it is likewise admissible that purpose and chance are incompatible, i.e. (3) $P.K = 0$.
Therefore

$$C.T.\bar{K}(P+\bar{P})+\bar{C}.\{T.\bar{K}.(P+\bar{P})+\bar{T}.(P.\bar{K}+\bar{P}.K+\bar{P}.\bar{K})\} = 1. \quad (3.41)$$

Consequently, if purpose and chance be such that although both cannot be; yet neither might be, as in the case of a mechanical order of Nature that was neither purposeful nor due to chance, but merely regulated by laws of causation; then certainty, though it implies truth and eliminates chance, does not necessarily imply a purposive intelligence. Unless therefore truth involves a rational purpose, or we fall back upon the ontological argument, could we assert that a purposive intelligence was implied in the feeling of certainty.

(2) Some philosophers, Sir Isaac Newton for instance, regarded the absence of chance as an indication of the existence of a Divine purpose as the only alternative, i.e. that $\bar{P}.\bar{K} = 0$; so that

$$C.T.\bar{K}.P+\bar{C}.\{T.\bar{K}.P+\bar{T}.(P.\bar{K}+\bar{P}.K)\} = 1. \quad (3.42)$$

Here it is still possible that truth and chance might be both absent, i.e. that truth and chance are not absolutely alternatives. In which case there might be an irrational purpose which, though not due to chance, was nevertheless not the direct medium of truth. We should require to have that $\bar{T}.\bar{K} = 0$ in order to eliminate this.

(3) If C = feeling of certainty,
t = compatibility with truth,
w = chance working of nature,
P = a purposive intelligence,
r = rational processes,
C_1 = self-conscious apprehension,

then if we accept the very obvious premisses (1) $C\bar{t} = 0$; (2) $w.P = 0$; (3) $t.\bar{r} = 0$; (4) $r.\bar{P} = 0$; (5) $r.\bar{C}_1 = 0$; (6) $C_1.\bar{C} = 0$, we easily obtain that

$$C_1.r.\bar{w}.P.C.t+\bar{C}_1.\bar{r}.\bar{C}.\bar{t}.\{w.\bar{P}+\bar{w}.(P+\bar{P})\} = 1. \quad (3.43)$$

The inferences that may be drawn from this are obvious. Thus although the feeling of certainty necessarily implies a purposive intelligence, yet, on the other hand, the premisses do not imply that a purposive intelligence would necessarily, though incompatible with mere chance, be either rational or self-conscious (in the human sense, in which alone this could be said). For a purposive intelligence, if it refers to a Perfect Being, would of course be rational and self-conscious in the highest and best sense.

If we combine this latter equation with the previous one (3.42) we obtain, putting $t = T$, and $w = K$,

$$C.C_1.r.\bar{w}.P.T+\bar{C}.\bar{C}_1.\bar{r}.\bar{T}.(w.\bar{P}+\bar{w}.P) = 1, \quad (3.44)$$

the inferences from which are still the same as before, with this important difference, that whereas in the previous result an unconscious and irrational process might or might not have a purpose, even if it be not the mere outcome of chance, since an unconscious and irrational process might be the result neither of purpose nor chance, and might in fact be mechanical and worked according to certain laws, and not through mere chance: but in the latter proposition there is no alternative between chance and purpose; this of course is due to the fact that in the deduction of the equation (3.42) we put $\bar{P}.\bar{K} = 0$; or $\bar{P}\bar{w} = 0$.

(4) Put M = mind, and use the other symbols in the same sense as before; with the following premisses:

(1) $r.\bar{M} = 0$; All rational things are Mind.
(2) $T.\bar{M} = 0$; Truth is mind.
(3) $M.\bar{P} = 0$; Mind is a purposeful thing.
(4) $r.\bar{T} = 0$; Rational things are truth, i.e. reason is truth.
(5) $T.\bar{r} = 0$; Truth is a rational thing.

Then $$M.P.(T.r+\bar{T}.\bar{r})+\bar{M}.\bar{T}.\bar{r}.(P+\bar{P}) = 1. \quad (3.45)$$

Hence to deny a purpose is to deny reason and truth, as well as mind itself. Yet from the premisses such a purpose might or might not be of the nature of mind. And even if it be mind it might or might not constitute a rational purpose. Whilst if it be not mind it would obviously not be rational, although it might exist and manifest a purpose, as though it were rational. But if there be such a thing as truth, then there must be both reason and purpose, no less than mind their necessary correlative.

Now if we combine this with the previous equation (3.44) we obtain a more detailed and precise statement of this proposition. Thus:

$$M.P.\bar{w}.(T.r.C.C_1+\bar{T}.\bar{r}.\bar{C}.\bar{C}_1)+\bar{M}.T.\bar{r}.\bar{C}.\bar{C}_1(\bar{w}.P+w.\bar{P})=1. \quad (3.46)$$

Hence all mind though purposeful is not necessarily conscious; and purpose, on the other hand, is not necessarily mind, though it is not mere chance and accidental.

It will be observed that P is used here for a purposeful intelligence and not necessarily for a Perfect Being. The statement $M.\bar{P} = 0$; is obviously not true if P stands for perfection. If the proposition (3) above be omitted, and it be admitted that mind might or might not have a purpose, then the full statement of the case becomes

$$T.r.M.(P+\bar{P})+\bar{T}.\bar{r}.(M+\bar{M}).(P+\bar{P}) = 1. \qquad (3.47)$$

Hence truth and reason are concomitant, implying mind with or without purpose. The two latter, however, do not necessarily imply the two former; and there may be mind with or without a purpose; and purpose with or without mind, only feigning to be so, and in each case irrational and devoid of truth as an essential accompaniment of its performances. The absence of purpose therefore would not mean the absence of mind, or vice versa; but it is possible that there might be neither, although it would follow that neither truth nor reason could manifest itself in such circumstances.

We may now consider another aspect of the problem.

Let A = the Absolute,
a = the transitory,
P_1 = a Perfect Being,
C_1 = self-conscious apprehension.

Then (1) The Absolute is not transitory, i.e. $A.a = 0$,
(2) The perfect is Absolute, i.e. $P_1.\bar{A} = 0$.

Then we have

$$A.\bar{a}+\bar{A}.(a+\bar{a}) = 1,$$
$$A.(P_1+\bar{P}_1)+\bar{A}.\bar{P}_1 = 1. \qquad (3.48)$$

Therefore $A.\bar{a}.(P_1+\bar{P}_1)+\bar{A}.(a+\bar{a}).\bar{P}_1 = 1.$

Now if self-conscious apprehension can be predicated of a Perfect Being as it obviously could and should be, then (3) $P_1.\bar{C}_1 = 0$. Moreover, since obviously all that is capable of self-conscious apprehension is part of the Absolute (4) $C_1.\bar{A} = 0$.
Therefore

$$C_1.A.\bar{a}.(P_1+\bar{P}_1)+\bar{C}_1.\bar{P}_1.\{A.\bar{a}+\bar{A}.(a+\bar{a})\} = 1. \qquad (3.49)$$

The Absolute is therefore not necessarily self-conscious or perfect.
The self-conscious is not transitory.
The Absolute if not self-conscious is imperfect.

The transitory is not the Absolute, and it is imperfect and not self-conscious.

The non-transitory or permanent is not necessarily the Absolute. The equation may also be written

$$A.\bar{a}.\{C_1.(P+\bar{P})+\bar{C}_1\bar{P}\}+\bar{A}.\bar{C}_1.\bar{P}.(a+\bar{a}) = 1. \quad (3.50)$$

So far it is clear that the permanent, i.e. the non-transitory, is not necessarily the Absolute.

We may, following a similar line of argument, arrive at results that can be combined with the preceding ones, thus:

If G = The Supreme Being which is All-Wise, All-Good, and All-Perfect, $= P$,

T = truth,

x = Material things,

w = things caused,

d = error or delusion,

then, since 'God is uncaused', (1) $G.w = 0$.

'Truth is uncaused', (2) $T.w = 0$.

'Truth is not error; and there is no alternative between Truth and error or delusion', (3) $T.d = 0$, $\bar{T}.\bar{d} = 0$, i.e. $T.\bar{d}+\bar{T}.d = 1$.

Further, 'All that is material is caused,' (4) $x.\bar{w} = 0$.

'"Error or delusion cannot be uncaused," i.e. "They are caused",'

$$(5)\ d.\bar{w} = 0.$$

Then obviously, putting $G = P_1$,

$$x.\bar{G}.\bar{T}.d.w+\bar{x}.\{G.T.\bar{d}.\bar{w}+\bar{G}.(T.\bar{d}\bar{w}+\bar{T}.dw)\} = 1. \quad (3.51)$$

Hence all material things are delusions and not true. Truth is alone not error, and God is truth; but nevertheless truth is not necessarily God, at least so far as the argument goes.

Now if we combine these two results we obtain, assuming that

$$(6)\ d.A = 0; \quad (7)\ a.\bar{w} = 0; \quad (8)\ T.\bar{A} = 0:$$

$$x.\bar{G}.\bar{T}.d.w.\bar{C}_1.\bar{A}(a+\bar{a})+\bar{x}.[G.T.\bar{d}.\bar{w}.A.\bar{a}.C_1 \quad (3.52)$$
$$+\bar{G}.\{T.\bar{d}.\bar{w}.(C_1+\bar{C}_1).A\bar{a}+\bar{T}.d.w.\bar{C}_1\bar{A}.(a+\bar{a})\}] = 1.$$

It follows from this that all conscious things are not transitory but permanent, uncaused and immaterial; although it is possible for the permanent to be caused whether material or immaterial. The point is, truth and consciousness become one.

The material may or may not be permanent; but it is not conscious. The immaterial may be transitory. The Absolute, though permanent and true as well as uncaused, may not be conscious.

§ 3

We have assumed throughout these considerations that the law of contradiction and also that of the excluded middle term hold good. Now *the continuity that the Dialectic method of Hegel implies is of such a nature that most of these conclusions would have to be modified if it were admitted.*

When we come to consider the influence of chance, this is the haphazard play of atoms, or their more modern equivalents. Whether as conceived in the Wave-Mechanics or the Monadology, it would appear at first sight that given a sufficient time, an infinite time, we should find the Universe as it is. The purpose that we note could frankly have been itself evolved by the survival of stable groups of atoms, which, once formed by chance, would of themselves tend to persist [Professor S. Alexander in his *Space, Time, and Deity*, starting with a Space-Time Continuum, concludes that this Deity-like emergence constitutes the highest form of development. But if the time be infinite, He would long since have attained perfection, and though some groups of beings may still remain and perhaps ever remain imperfect, others such as Deity itself would have gained infinitely in the progress of development. So that we should find Perfection now, much as the old philosophers demanded. That Space-Time should be continuous requires that, between two events there is always an intervening event; however, we are still encountered by the same difficulties as with space or time separately. Alexander regards space and time as mere appearances owing to the inconsistency involved by continuity in either; but it being assumed that Space-Time is the reality, I must confess that I fail to see why the latter is not open to the same objections as the former, if there is continuity between events in a Space-Time *continuum*.]

Anyway, chance and purpose are not necessarily incompatible in an infinite time and even truth itself might be the result of the formation of stable groups of ideas in an infinite time. But though truth, reason, and purpose should have evolved or emerged by chance from a chaos of ideas in an infinite time, their presence in finite time compels us to regard them for finite purposes as the bases of our finite being. This fusion of ideas through emergence or even by chance forms the starting-point as a monad in the subsequent course of events. It comes to this, that the Creator was self-created. To call it chance throughout is to ignore the Infinite: further, two are ultimately one forms an antinomy of thought. Nevertheless chance would precede purpose, if purpose

could be evolved by chance as it no doubt could be, had chance operated for an infinite time. We cannot get over the contradiction except in the transcendental idea of orders of infinity. We are, I am convinced, ultimately driven to the similar conception of an Infinite Being as preceding an infinite order of infinities. This surely was the Cartesian idea of Perfection, and places reason above chance.

Cantor's theory of infinite numbers where $\aleph$ = number of induction numbers, i.e. numbers which obey mathematical induction starting with 0; otherwise called cardinal numbers, gives

$$\begin{aligned} \aleph_0+1 &= \aleph_0 \\ \aleph_0+n &= \aleph_0 \text{ when } n \text{ is an induction number.} \\ \aleph_0^2 &= \aleph_0 \end{aligned} \qquad (3{\cdot}53)$$

[See Lord Russell, *Introduction to Mathematical Philosophy*, p. 86.]

Galileo thought that the apparent contradiction between the number of cardinal numbers and the number of their squares could be got over, because the squares are only some of the cardinal numbers. But the point is that it is a one-one relation and that every cardinal number has its square. Hence the number must be the same. As Professor J. W. Scott has indicated (see 'Mathematico-physical Speculation and Philosophy', *Proc. Aristotelian Soc.*, Nov. 26, 1923): 'If we have a collection with an infinite number of terms in it, then whatever other infinite collection we care to take, we can produce a number of this one to match every number of that, so far as we care to go. This, however, though it means that each collection has the same number in it, does not appear to mean that the infinite collection *A cannot be any greater* than the infinite collection *B*. It can be greater, for *B* may be only a part of *A*. You can match every natural number with its square, however far you care to proceed with the list.' The point is that when you get to the infinite the difference between the two infinite numbers becomes nil, $\aleph_0^2 = \aleph_0$, just as in the case of identity of a unit with itself, the parts of which may be regarded as infinite, ordinary arithmetic no longer holds, $n = 2n = n^2 = n^m$, two becomes merged into one. Arithmetic holds only for the finite parts.

And conversely in regard to a point or moment. Multiplicity can 'emerge' from unity, and two or more appear where there was only one, if only an infinite collection of parts that make up a point could by a similar one-one relationship emerge into finite entities. It is no more inconceivable according to the Cantorian definition of infinity, that two should merge into one in the infinitely large, than

that two should emerge from one in the infinitely small. Identity becomes plurality, and diversity, for there is continuity between them.

§ 4

If Perfection is the goal of Purpose in infinite time then wherever beauty is, there also is God; for beauty is ideal perfection, but it must be beauty pure and undefiled. For evil is but the principle which tends to destroy all that is good and beautiful. God is the inherent nature of all being, metaphysically speaking, 'The Principle of Concretion'. This principle is without digression then none other than that which Tennyson had so well expressed in his oft-quoted lines, too beautiful to be ever regarded as a tag, and for this reason I venture to repeat them:

> 'Flower in the crannied wall,
> I pluck you out of the crannies,
> I hold you here, root and all, in my hand,
> Little flower—but *if* I could understand
> What you are, root and all, and all in all,
> I should know what God and man is.'

'This concretion' or hypostatical abstraction lies in the implication that all possible knowledge of existing things is involved in the knowledge of the particular: in fact the unification of the many in the one; so that all being is brought to a focus in the particular, which actually implies that the infinite is the finite as applied to identity from consideration already adduced. At first sight this may seem to be nothing more than the Aristotelian idea of the universal, which the Schoolmen designated as *universalia in re*. More accurately perhaps Hegel's 'notion' as distinct from the mere 'idea' which admits of negation; for the 'notion' is the higher synthesis of the idea and its opposite: in fact the unity between contradictories. It is hard to say where exactly Dr. Whitehead's conception of the Deity discriminates between these or overcomes the real difficulty of reconciling the one with the many; the real problem of philosophy. For Pringle-Pattison has well said:

'The question of universals, though fully discussed, no longer forms the centre of speculation (among the schoolmen from Albertus Magnus onwards). The age of Scholasticism presents, indeed, a substantial unanimity upon this vexed point, maintaining at once in different senses, the existence of the universals *ante rem*, *in re*, and *post rem* (the Platonic, Aristotelian, and Nominalistic views respectively). Albert and Aquinas both professed the moderate Aristotelian Realism which treats *genera* and *species* only as *substantiae secundae*, yet as really inherent in the individuals, and constituting their *form* or *essence*. The universals therefore have no existence, as universals, in *rerum natura*; and Thomas endorses, in this

sense, the polemic of Aristotle against Plato's hypostatized abstractions. But the Augustinian sense of ideas immanent in the Divine mind, the universal *ante rem* (the Platonic universal), may well be admitted as possessing real existence. Finally, by abstraction from the individual things of sense, the mind is able to contemplate the universal apart from its accompaniments (*animal sine homine, asino, et alias speciebus*); these subjective existences are the *universalia post rem* of the Nominalists and Conceptualists. But the difficulties which embarrassed a former age in trying to conceive the mode in which the universal exists in the individual reappear in the systems of the present period, as the problem of the *principium individuationis*.'

The 'what-ness' and the 'this-ness' of every idea; the universal and the particular once more in a disguised form.

In what way precisely the principle of concretion helps us to realize the unity of all things and the presence of God in all that is beautiful as divine, this indeed is the problem of philosophy that Dr. Whitehead attacks and answers in his own admirable way both as a mathematician and a philosopher. The question of the personality of God which Lotze and Ward emphasized, and upon which Hegel did unquestionably though in a different manner insist, is the Divinity of Christ; this very concretion or incarnation of God in man must surely enter as the unifying principle in Dr. Whitehead's conception of the Deity. A simple and concise exposition of this concept is given by Professor Henry Nelson Wieman in the *Hibbert Journal*, July 1927. He shows that Whitehead's idea of God is not pantheistic:

'firstly; because he represents God not as the totality of all being, but as a certain dominant and sustaining character or order of all being. Secondly; because God does not include evil, evil being a certain antagonistic character or disorder that appears in the universe. Thirdly; because God transcends the world of existence and is much more fully operative as the characteristic order of the realm of possibility, and in the realm of abstract forms generally, than in the world of actuality, although he is the constitutive principle of concrete order in both.'

That the infinite is continuous with the finite; whether the infinite is self-conscious and intelligent; these are questions which eminent metaphysicians like Dr. McTaggart do not admit. To Leibnitz and the greater developments of his system as worked out in the *Microcosmos* of Lotze, that immanence seems almost self-evident, and the self-consciousness or Ego of the infinite the most real and most true of all facts. Whatever defects may be found with this aspect of the question, it is that which appeals most forcibly to the practical reason as the solution of the incongruities of Nature. Poetically we may observe in *Faust* the human interpretation of it. But not in Goethe alone does it form the key-note

to character and the mystery of life. Tolstoy (a contemporary of Lotze's rather than of Goethe's) follows with the same idea running through his works. It will be remembered how Pierre tells us that

'The joyous feeling of freedom, that perfect inalienable freedom inherent in man, a realizing sense of which he had for the first time experienced at the first halting place, when he was carried away from Moscow [where he had been taken prisoner by Napoleon's troops and fallen ill], filled his soul during his convalescence. He was amazed that this inner freedom, independent of all external circumstances, now, as it were, surrounded him with an excess, with a luxury of external freedom. . . . The very thing which he had formerly tormented himself about, and constantly sought in vain—an object in life—now no longer existed for him. That long-sought-for object of life was not merely absent by chance for the time being, but he felt that it did not exist and could not exist. And it was precisely this absence of an object in life which at this time constituted his happiness. He could have no object, because now he had a faith—not a faith in any rules, or creed, or dogmas, but *faith in a living everywhere perceptible Infinite.*

'Hitherto he had sought for It in objects which he had set for himself. This searching for the object was only the seeking for God, and suddenly, during his captivity, he had learned, not from words, not from reasoning, but from his immediate consciousness, what his old nurse used long, long before to say, that God was here, there, everywhere. He had learned, during his captivity, that God in Karatayef [a fellow prisoner and common soldier, but something of a genius, whilst Pierre was an accomplished gentleman and a millionaire]—he had learned that God in Karatayef his companion was more majestic, endless, and past finding out, than in what we are accustomed to call the Architect of the Universe. He had a similar experience to that of the man who should find under his very feet the object of his search, when he had been straining his eyes in looking at a great distance. All his life long he had been looking away over the heads of the surrounding people while all the time there had been no need to strain his eyes, but merely to look straight ahead.

'He had not been able hitherto to see the Great, Incomprehensible, the Infinite in anything. He had only felt that It ought to be somewhere, and he had searched for It.

'In all that was near and comprehensible, he had seen only what was limited, the narrow, finite, meaningless. He had provided himself with a mental telescope, and looked out into the distance, yonder, where this narrow, finite object, concealed in the murky distance, seemed to him great and infinite, simply because it was not clearly seen.

'In this way European life, politics, philosophy, philanthropy, had presented themselves to him. But at the very moment when he had accounted himself most weak, his mind had leapt forth into that same distance, and then he had seen how small and narrow, how finite and meaningless, it all was.

'Now, however, he had learned to see the Great, the Eternal, and the Infinite in everything, and therefore, naturally, in order to see it, in order to enjoy the contemplation of it, he had thrown away his telescope, through which he had till then been looking over men's heads, and joyfully contemplated the ever-changing, incomprehensible, and eternal life around

him. And the more closely he looked, the more serene and happy he became.

'The terrible question which hitherto had overturned all his mental edifices—the question *why*—no longer tormented him. His mind was always ready with the simple answer, *Because the Infinite is*, that God without whose will not a hair falls from the head of a human being.' (Tolstoy, *War and Peace*, vol. iv, p. 222.)

The infinite in the smallest as well as the greatest of things; the continuity which renders all things one and comprehensible, such is God. That infinite must be self-conscious, else it would not be infinite and perfect. This is Hegel's Absolute Idea.

Commonplace as it may seem to him who has not realized its actual meaning, it is in fact the greatest of all truths to the person accustomed to dwell upon these matters. 'Nominal', but 'real', personal, human sense which draws forth the sublime powers of the mind; not merely of the poetic or dramatic faculties, but of full manhood. To such a one is the experience of Pierre, a vivid and true representation of the facts of life. In it the mind takes in as a whole the meaning of the Infinite; without, it is true, perceiving the particulars of the wondrous structures. Only the Omniscient Intelligence can grasp its own contents in detail. But even the finite intellect is not deprived of a glimpse—feeble and insignificant as that glimpse may be—of the faint glimmer of reality—of the immeasurable and yet continuous fact of existence, without beginning and without end, as it must always appear to the questioning intellect and reflective mind, in the earnest contemplation of truth. The doctrine or theory of immanence, rendered familiar in *Faust* by Goethe, has its basis in the practical reason for all men who dwell upon the great enigma of being. By the development of culture and the advancement of knowledge is progress in this direction to be achieved and a clearer conception to be attained.

CHAPTER X

THE PHENOMENAL WORLD AS DISTINCT FROM THE REAL

THE striking feature in Hegelianism is its self-sufficiency. The positivist claims to see only one side and to see that side as clearly as it can be seen to make it positive. The agnostic does not claim to see either side clearly enough to be sure of anything and is overwhelmed by the uncertainty of his insufficiency. But the Hegelian not only sees it partially but sees it whole. If we would but bear in mind the geometrical parallel the difficulty in Hegel might be lightened if not removed. Identity and difference are the same but different aspects of the same and yet different things.

In the algebraic treatment of the subject the introduction of imaginary classes involving functions directly or indirectly containing $\sqrt{-1}$, a potential class not actually existing, but admitting on expansion of realization or emergence, enables us to express the remarkable appearance in time, from apparently nowhere, or nothing, of types that had lain hidden as it were, as permanent possibilities of existence. This is the most striking characteristic of the method; and its development entails the very essence of the idea as the all in all and the all containing all. When we consider the expansion of the function that expresses the complete universe of discourse,

$$(x_1+\bar{x}_1).(x_2+\bar{x}_2).(x_3+\bar{x}_3)\ldots(x_n+\bar{x}_n) = 1.$$

x_1, x_2, x_3,..., x_n being, as usual, any classes whatsoever, then since we could regard each of these as made up of a real and an imaginary element thus $x_n = a_n+ib_n$, where $i^2 = -1$, it follows that as n is increased new combinations of classes will arise, apparently from nowhere. If the element of *time* be taken as a factor, then obviously, if we carefully bear in mind the logical signification of the equation and the actual sense in which it is intended to be interpreted, emergence occurs. For taking $x_n = t^n$, and $t^n = t$, the roots of unity when substituted in the equation of the universe of discourse can be (if the time be regarded, as it should be, in continuous progression) the means of unexpected emergence in the expansion of the factors of time to infinity. If time then be one of the factors, it obviously, sooner or later, must develop all the terms that in the earlier stages lay hidden owing to their association with the imaginary terms or roots of unity; but as time and events unfold, the full evolution of the idea is realized in the emergence of the

possibilities as the Divine Idea of Logical Truth. This perhaps is the Hegelian conception of the Idea. But Hegel gave no adequate explanation of the mode of its development or the emergence of new forms in the progression of time. Nor indeed did he give any adequate clue to the repetition in time of many forms, such as the reproduction of living types in the time series. This is apparent in the manner of the incommensurable classes to which I have already directed attention. Such emergence would constitute abrupt variations, as distinct from the commoner variations arising from the simple identity types. These latter, however, in passing from one scale to another in space, may, as has already been shown, be transformed from continuous to abrupt variations, or mutations.

In dealing with the applications of this idea and principle of logic to the phenomena of Nature and of thought, it must be borne in mind that there is, as Sir John R. Seely in his appreciation of Goethe has so well said, 'nothing too high or too deep for it; too broad or too narrow'; too far removed or too near; too large or too small; too great or too insignificant or indeed too valuable or too worthless to exclude aught else. And we might add too hideous or too beautiful; too wicked or too good; too detestable or too lovable; too coarse or too refined; perhaps even too worthy or too insincere; too gracious or utterly lacking in good nature. Nothing in fact that does not include or imply its opposite or even its contradictory. For Art embraces all reality and every possibility; and truth and error like reality and unreality are included in the conception of the whole, as the totality. This from its very nature must include everything and exclude nothing. In this way and in this way alone is the essential as distinct from the non-essential never lost sight of. He who seeks truth without clearly bearing in mind the presence of error will never attain the object of his quest. And in the gradual elimination of error, truth ultimately emerges sublime, triumphant, and supreme, whilst the process of selection takes place by the sifting of the suitable ideas in the selective process of time. This selection is guided somewhat by the emergence of new types, ever in accordance with the rational laws of algebraic thought which in most cases remains indeterminate.

All the phenomenal worlds represented by the equation $x^n = x$ are merely appearances, corresponding to negative or, worse still perhaps, imaginary values of x, except that one for which x is 1. The law of self-contradiction, as we have seen, applies only to the last; all the others involving either -1, or $\sqrt{-1}$, which in the course of time unfold and emerge into the actual idea of reality. It is unnecessary to repeat further what has already been said upon

this point. Emergence of all possible types must sooner or later arise; but their persistence depends upon the previous history and the types that shall have survived through their stability and adaptability to the circumstances of their environment. So that a directive tendency is in time evolved. Now, in so far as evolution in one scale of being in space, by continuous variation as the Darwinian theory requires, leads to discontinuous variations in a higher scale of being also in space; as in the passage from the atomic to the organic world; so would the effects of that evolution already attained in the infra or atomic world reflect itself as it were in the organic world of our being in the scale of our habitation; and though manifesting itself, mostly as discontinuous variations or mutations, exhibit the signs of a design or purpose which the evolution in the infra world of the atomic had already predetermined. This is a point that Darwin, not being a physicist, was apt to overlook, and in fact did overlook, to the eternal confusion of his critics and followers alike. The principle once recognized, continuous variations in one scale can give rise to discontinuous variations in the other. The development which takes place in the individual from the microscopic germ to the fully matured organism is itself possible only because of such an evolutionary process in the infra or atomic world. Were it not for the directive force and stamp of character there acquired, the subsequent development of the cell into an organism would not be sufficiently stable to take place. This process of evolution must likewise be conceived as having occurred in an endless series from scale to scale in the gradations of space. And if not in an infinite series yet it should be regarded as at any rate not having commenced in the atomic world of chemistry; for as shown elsewhere (p. 372) the mechanism involves more units than the cell contains.

§ 1

In general, as we have seen,

$$x.\bar{x} = 0, \text{ or } x.(1-x) = 0; \text{ i.e. } x^2 = x;$$

also $$(x+\bar{x}).(x+\bar{x}) = 1; \text{ i.e. } (x+\bar{x})^2 = 1. \qquad (3{\cdot}54)$$

But if $$x\bar{x} \neq 0 \quad x^2+x.\bar{x}+\bar{x}.x+\bar{x}^2 = 1.$$

This Boole overlooks. Now since $x.\bar{x} \neq 0$, it cannot be said that 'Everything is either x or not x, as some things would be both x and not x; the law of contradiction not holding true for the particular class of things, ideas, or propositions. There is in this case continuity between x and not x; through the class that is in

transition between them, and which is at once both x and not x. This can also be written $\bar{x}.(1-\bar{x})$; that is the class that is neither x nor not x.

The stages of transition from the future to the present, or from the present to the past are cases in point, in which a contradiction appears to exist; both space and time being 'forms of intuition' and not 'things-in-themselves'; mere appearances, as distinct from realities.

Thus in general $x_1+\bar{x}_1 = 1$, $x_2+\bar{x}_2 = 1$, $x_3+\bar{x}_3 = 1$, etc., if the law of contradiction holds; i.e. provided that we are dealing with realities and not with appearances. But when we have to contend with phenomena rather than with noumena, as is in fact the case with the appearances of ordinary life in Nature, then the proper form of this equation is not

$$(x_1+\bar{x}_1).(x_2+\bar{x}_2).(x_3+\bar{x}_3)\ldots(x_n+\bar{x}_n) = 1$$

but
$$(x_1^2+x_1\bar{x}_1+\bar{x}_1^2).(x_2^2+x_2\bar{x}_2+\bar{x}_2^2)\ldots(A+\bar{A}) = 1, \qquad (3{\cdot}55)$$

where A is what is absolutely true, so that

$$A.\bar{A} = 0. \qquad (3{\cdot}56)$$

But for absolute truth we should also have

$$x_1\bar{x}_1.A = 0;\ x_2\bar{x}_2.A = 0, \text{ etc.}$$

Therefore
$$(x_1+\bar{x}_1).(x_2+\bar{x}_2)\ldots(x_n+\bar{x}_n).A+\bar{A} = 1, \qquad (3{\cdot}57)$$

where $\bar{A}$ covers all the non-linear factors.

Hence all the Boolian equations postulate the factor A, i.e. that we have to deal with the absolute truth and not with an approximation thereto; as is in fact the case with the paradoxes of philosophy.

In Boole's system, though $x^2 = x$ is admitted as a necessary law of thought, yet $x^3 = x$ has no meaning, except in so far as it satisfies the values $x = 0$ and $x = 1$; but $x = -1$ for him had no signification whatever. The factors of the equation $x.(x^2-1) = 0$ being $x.(x-1).(x+1) = 0$, the law of identity

$$x.x.x.x...x = x, \text{ or } x^n = x, \qquad (3{\cdot}58)$$

does not necessarily imply that of contradiction and excluded middle term of ordinary logic. Here perhaps we find ourselves in conflict with reason itself, a difficulty which Boole gets over in the ordinary way by ignoring the issue. He admits that if the laws of thought had been other than they are, the equation $x^3 = x$ might have had some meaning. But this is just what it has in the Hegelian logic of Dialectic.

The equation for identity $x^n = x$ can be written thus:

$$x.(1-x).(1+x+x^2+x^3+x^4+\ldots+x^{n-2}) = 0.$$

The roots of $1+x+x^2+x^3+x^4+\ldots+x^{n-2} = 0$ (3·59)

would satisfy the identity $x^n = x$, even if they did not comply with the condition of non-contradiction as expressed by the equation

$$x.(1-x) = 0.$$

Hence every one of the $(n-2)$ roots gives a continuity between x and $\bar{x}$. The fact that they may be imaginary does not affect the reality of the result as manifested in the resultant equation; any more than the use of the imaginary $i = \sqrt{-1}$ as a factor of time in the theory of relativity hinders the development of the idea of a 'Space-Time Continuum'. The employment of the algebraic imaginary is already so familiar that emphasis need hardly be laid on its importance.* Its use in logic, however, is somewhat novel; and it is not to be wondered at that in Boole's day it should have been altogether overlooked. We owe its inception as a useful factor to Sir William Rowan-Hamilton, the Irish Newton of T.C.D. in the invention of Quaternions in 1843. Boole's work came shortly afterwards: *The Mathematical Analysis of Logic* in 1847, and the *Laws of Thought* in 1854. It is not surprising therefore that he should not have perceived the utility of the Hamiltonian conception of the imaginary in its relation to his own work. The resemblance is in the use of the imaginary $\sqrt{-1}$.

If $n = 3$ we have $x.(x^2-1) = 0$, or $x.(x-1).(x+1) = 0$; so that x may have any of the three values 0, 1, or -1; and in the latter case it is not at all necessary that $x.(x-1) = 0$ should hold good; hence $x^2+x.\bar{x}+\bar{x}^2 = 1$;

so that there may be perfect continuity between x and $\bar{x}$ as in the dialectic of apparent truth in the Hegelian logic.

The logical interpretation of $x = -1$, in the algebra of logic or in fact of any of the roots of the equation

$$\sum_{n=2}^{n=n} x^{n-2} = 0, \qquad (3.60)$$

is as difficult as the interpretation of $\sqrt{-1}$ in Physics; yet both play a most important part in the world of phenomena: one in the realm of thought, the other in the realm of Nature.

If $n = 4$, then $x.(x^3-1) = 0$, and $x.(x-1).(x^2+x+1) = 0$, the roots of the quadratic factor being

$$x = \frac{-1 \pm i.\sqrt{3}}{2}, \quad \text{where} \quad i = \sqrt{-1}.$$

* It was introduced as a meaningless symbol by the Italian Cardan in 1545.

For $n = 5$ we have $x.(x^4-1) = 0$,

or $$x.(x-1).(x+1).(x^2+1) = 0,$$

so that $x = \pm i$ are possible solutions.

For $n = 6$, obviously

$$x.(x-1).(x^4+x^3+x^2+x+1) = 0.$$

For $n = 7$, we obtain

$$x.(x-1).(x^5+x^4+x^3+x^2+x+1) = 0.$$

This readily breaks up into the factors

$$x.(x-1).(x+1).(x^2+x+1).(x^2-x+1) = 0,$$

giving $$x = \frac{\pm 1 \pm i.\sqrt{3}}{2}.$$

For $x = 9$, we obtain

$$x.(x-1).(x+1).(x^2+1).(x^4+1) = 0;$$

and so on *ad infinitum*.

In general when the laws of contradiction and excluded middle apply, as in ordinary logic, we have

$$(x)+(1-x) = 1;$$

which may, however, be also written

$$(-x)+\{1-(-x)\} = 1. \qquad (3.61)$$

The ordinary laws of thought would therefore apply to the negative of a class; and $(-x).\{1-(-x)\} = 0$, provided $x = -1$.

In this case $x.(1-x) \neq 0$.

Hence if x is a class of phenomena with $x(1-x) \neq 0$ and $x = -1$, it would appear that $(-x)$ might be regarded as though it were the noumena of these phenomena; for if the class would not obey the laws of thought the negative of the class itself as a noumenon would do so.

It must be borne in mind, however, that the negative of a class is not the contradictory nor the contrary, but something entirely transcendental in thought of algebraic significance, like i, j and k in vector analysis, where a rotation is effected through two right angles, but it is not imaginary.

Thus the negative of the world of phenomena may be the real world of pure transcendental thought in which alone the laws of thought hold, whilst the world of phenomena is subject to the dialectic of continuity and contradiction. Hence if x is a phenomenon $(-x)$ may be regarded as the noumenon. We can form no mental picture of what this may actually mean.

If we assume that $x.\bar{x} \neq 0$, i.e. that x may be not-x, we have, of course,

$$x^2+x.\bar{x}+\bar{x}^2 = 1;$$

since the actual numerical coefficients of the terms do not enter into account, we have where continuity is to be found

$$x^n+x^{n-1}\bar{x}+x^{n-2}.\bar{x}^2+ \ldots +x^2.\bar{x}^{n-2}+x.\bar{x}^{n-1}+\bar{x}^n = 1. \quad (3.62)$$

But if ρ be the real, and ρ the not-real, then $x.\bar{x}.\rho = 0$; likewise $x^n.(\bar{x})^m.\rho = 0$; we have

$$\rho(x^n+\bar{x}^n)+\bar{\rho}.(x^n+x^{n-1}.\bar{x}+\ldots) = 1. \quad (3.63)$$

This is assuming that $\rho.\bar{\rho} = 0$, i.e. that there are no degrees of reality and that everything is either real or not real. Similarly as regards the truth. But Hegel supposed that there are degrees of truth, which are not truth at all but only appearances or delusions; phenomena not noumena. There is perfect discontinuity between ρ and $\bar{\rho}$; though Hegel supposes that by higher and higher synthesis it is possible to attain to the Absolute. This can only be if there is continuity between the real and the unreal, and would be the case not if $\rho^n = \rho$ only; but when

$$\rho^n+\rho^{n-1}+\rho^{n-2}+\ldots+1 = 0; \quad (3.64)$$

that is if there is continuity between ρ and $\bar{\rho}$, as the roots of this equation would imply that

$$\rho^n+\rho^{n-1}.\bar{\rho}+\rho^{n-2}.\bar{\rho}^2+\ldots+\rho^2.\bar{\rho}^{n-2}+\rho.\bar{\rho}^{n-1}+\bar{\rho}^n = 1. \quad (3.65)$$

In this case we cannot make the laws of contradiction and excluded middle term the tests of reality, but must confine ourselves to the law of identity. The real then merges into the unreal by a strict continuity. These conceptions, although purely algebraic and apparently inapplicable to logic, do in fact admit of an important signification.

Obviously we have to deal here with the overlapping of an idea and its contradictory; and again with the overlapping of identities! In the Platonic Theory this is never overcome. Only in the algebraic expression of this phenomenon is any attempt at an explanation rendered possible.

If there are two classes x and y, and if $x.\bar{x} \neq 0$; $y.\bar{y} \neq 0$;

then $$(x^2+x.\bar{x}+\bar{x}^2).(y^2+y.\bar{y}+\bar{y}^2) = 1. \quad (3.66)$$

Only when $x.\bar{x} = 0$ and $y.\bar{y} = 0$ do we obtain

$$x.y+x.\bar{y}+\bar{x}.y+\bar{x}.\bar{y} = 1.$$

But if $x.\bar{x} \neq 0$ and $y.\bar{y} \neq 0$ and $x.\bar{y} = 0$, $x.y = 0$;

$$(x^2+\bar{x}^2).(y^2+y.\bar{y}+\bar{y}^2)+x.\bar{x}.(y^2+\bar{y}^2) = 1, \quad (3.67)$$

whereas if $x.\bar{x} = 0$ and $y.\bar{y} = 0$, we get

$$\bar{x}.(y+\bar{y}) = 1,$$

and therefore $x = 0$.

It is only if $x.\bar{y} = 0$ is not absolutely true that $x.y = 0$ is possible at the same time.

In general, then, if $x.\bar{y} = 0$, and at the same time $x.y = 0$, then

$$\bar{x}.(y+\bar{y}) = 1;$$

and if in addition $y.\bar{z} = 0$ and $y.z = 0$,

$$\bar{x}.\bar{y}.(z+\bar{z}) = 1;$$

and if $z.\bar{w} = 0$, and $z.w = 0$, we have likewise $\bar{x}.\bar{y}.\bar{z}.(w+\bar{w}) = 1$. So that ultimately the following must hold, that

$$\bar{x}.\bar{y}.\bar{z}.\bar{w}...\bar{\chi}\bar{\phi}\bar{\psi}(\Pi+\bar{\Pi}) = 1. \qquad (3.68)$$

Therefore

$$x = 0, y = 0, z = 0, w = 0, ..., \chi = 0, \phi = 0, \psi = 0,$$

$$\therefore \Pi+\bar{\Pi} = 1. \qquad (3.69)$$

Thus ultimately Π does not contradict itself; and its contrary is absolute negation. But in so far as the self-contradictory is not the Absolute it must be included in absolute negation; and therefore appearance is negation. The difficulty is that there is an absolute discontinuity between this negation and the Absolute. The Hegelian dialectic therefore never could attain by higher and higher syntheses to the Absolute as it professes to do, without taking a final jump across the impassable gulf that separates the two.

Thus the final synthesis is a discontinuous process where the law of contradiction holds.

It follows from the equation of identity $P = P^n$ that

$$P(P-1)(P^{n-2}+P^{n-3}+...+P+1) = 0. \qquad (3.70)$$

The roots of this equation are $0, 1, \omega, \omega^2, \omega^3, ..., \omega^{n-2}$, where

$$\omega = \cos\frac{2\pi}{n-1}+i\sin\frac{2\pi}{n-1},$$

so that $$\omega^{n-1} = \left(\cos\frac{2\pi}{n-1}+i\sin\frac{2\pi}{n-1}\right)^{n-1} = 1.$$

Similarly $$\omega^k = \left(\cos\frac{2\pi k}{n-1}+i\sin\frac{2\pi k}{n-1}\right)$$

$$(\omega^k)^{n-1} = \cos 2\pi k+i\sin 2\pi k = 1.$$

Now if $P \neq 1$ because $\bar{P} \neq 0$, since there are obviously some imperfect beings, and $\omega^1, \omega^2, ..., \omega^{n-2}$ all give contradictory $P\bar{P} \neq 0$,

either $P = 0$ or $P = \omega_1, \ldots, \omega^{n-2}$ does not represent reality. But we have already shown that for $P = P^3$, $R = -P$ gives $R(R-1) = 0$, and therefore the negative of the idea of P is the reality.

It would appear therefore that the reality, or noumenon corresponding to the idea of Perfection, need not be self-contradictory. The negative of the self-contradictory idea is the reality $R = -P$ corresponds to $P = \omega$. Therefore, if $P^3 = P$,

$$P(1-P)(1+P) = 0.$$

If $P = -1$, $\qquad P(1-P) \neq 0,$

but if $R = -P$, $\qquad 1-R = 1+P, \qquad R(1-R) = 0.$*

This has been already discussed on p. 161.

Moreover, since $P^{n-2}-1 = 0$,

$$1+\omega+\omega^2+\omega^3+\ldots+\omega^{n-2} = 0. \qquad (3.71)$$

Therefore $\qquad -\omega-\omega^2-\omega^3-\ldots-\omega^{n-2} = 1,$

i.e. $\qquad (-\omega)+(-\omega^2)+(-\omega^3)+\ldots+(-\omega^{n-2}) = 1.$

So that the totality is the *logical sum* of the various roots negatived either $-\omega$ or $-\omega^2$ or $-\omega^3$ ... or $(-\omega^{n-2})$,

i.e. $\qquad R_1$ or R_2 or $R_3 \ldots$ or R_{n-2},

where $\qquad R^n = R.$

The application of the Abelian equation to logic then opens up new possibilities.

When k is a prime number ω^k, ω^{2k}, ω^{3k} etc. have values other than unity, since $\omega^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$ and k is not divided by any whole number being a prime, it follows that ω^k, ω^{2k}, ω^{3k} etc. have values other than unity.

This is very important in the theory of emergence and variation, for from an identical unit, point, or moment, a variety may emerge. (See § 4, p. 251.)

* The bearing of this conception of reality or noumenon on Dirac's theory of negative energy may be of interest, but its discussion must be postponed.

CHAPTER XI

CONTINUITY AND DISCONTINUITY

§ 1

IN all subsequent discussions the factors $(s+\bar{s}).(t+\bar{t}).(S+\bar{S}).$ etc. enter, where s, t, S, etc., stand for space, time, substance, etc., as the categories of the understanding or forms of intuition, whether Kantian, Aristotelian, or otherwise, and $\bar{s}$, $\bar{t}$, $\bar{S}$ their contradictions. In order that the discussion should be exhaustive and precise, these factors must in every case be expressly stated or implied: otherwise the argument is inconclusive.

Thus Lord Russell's argument for the existence of objects without 'sense-data' is severely handled by Dr. Moore on the ground that Russell ignores the fact that two bodies in every respect similar could not be distinguished from each other, though in different places or at different times. This particular question is discussed by Kant at length in the *Kritik*. But it is not merely space and time that have to be taken into consideration. Substance likewise enters into the question. Two bodies might occupy the same place at the same time and yet be different though undistinguishable. For instance, the atoms of hydrogen and oxygen in the molecule of water. Two atoms may occupy the same space (see Clerk-Maxwell's article on the 'Atom', *Encyclopædia Britannica*, 9th edition).

Moreover, any number of bodies would be undistinguishable from each other even if they differed in quality whilst occupying the same space at the same time and having the same substance or chemical composition, as for instance in the case of isotopes. In every case the drawing of finer and finer distinctions involves the assertion of the suppressed factors always implied or understood in the equation of 'the universe of discourse'.

The subtlety of these distinctions gives a polish and refinement to the mathematical treatment of any and every subject that may be brought under consideration; and like the high-power or ultra-microscope reveals with ever-increasing clearness the hidden structures lying beneath the secret recesses of the understanding.

An interesting and elementary application of the method we are considering is the following:

If R = the real,
t = time,
P = the present,

p = the past,
f = the future,
then the most general statement which, in the absence of all limitations, includes every class of combinations of these terms is given by the equation

$$(R+\bar{R}).(t+\bar{t}).(P+\bar{P}).(p+\bar{p}).(f+\bar{f}) = 1.$$

Now since the present, the past, and the future are all included in the class time, we have

(A) $$P\bar{t} = 0, p\bar{t} = 0, \text{ and } f\bar{t} = 0.$$

Furthermore, as all time is either past, present, or future, it follows that (B) $$t\bar{P}\bar{p}\bar{f} = 0.$$

Again, if the present is real, but the past and the future are not so,

(C) $$P\bar{R} = 0, pR = 0, \text{ and } fR = 0.$$

Therefore

$$t.\{RP\bar{p}\bar{f}+\bar{R}\bar{P}.(pf+p\bar{f}+\bar{p}f)\}+\bar{t}\bar{P}\bar{p}\bar{f}.(R+\bar{R}) = 1,$$

We note that the premisses so far do not exclude the coexistence of the past and the future. But since obviously this also must be excluded, $pf = 0$; therefore

$$t.\{RP\bar{p}\bar{f}+\bar{R}\bar{P}.(p\bar{f}+\bar{p}f)\}+\bar{t}\bar{P}\bar{p}\bar{f}.(R+\bar{R}) = 1. \qquad (3.72)$$

Thus the real may not be in time at all.

Although the present is real, the future and the past are not so. Nevertheless we know that there is continuity between the past and the present; as also between the present and the future.

Now if there is continuity between p and P; as also between P and f; then must there be continuity between R and $\bar{R}$.

Therefore, either there is no such thing as the real or else it may be said with truth that everything is real. The premiss (C) would consequently need amendment.

Time therefore as a continuous creation from the apparently 'unreal' future to the real present, and as an annihilation of the real present into the apparently 'unreal' past, as memory, may be represented by the equation

$$t.\{RP+R\bar{R}.P.(p+f)+\bar{R}(p+f)\}+\bar{t}\bar{P}\bar{p}\bar{f}.(R+\bar{R}) = 1.$$

This may be written

$$t.(\bar{R}f+R\bar{R}Pf+RP+R\bar{R}Pp+\bar{R}p)+\bar{t}\bar{P}\bar{p}\bar{f}.(R+\bar{R}) = 1. \quad (3.73)$$

The term $R\bar{R}$ is like the imaginary $t\sqrt{-1}$ in the theory of relativity by the insertion of which the equation acquires a homogeneous form; and continuity between the future, the present, and the past is made at least algebraically manifest.

If now we alter the premiss C, so that

$$\left.\begin{array}{l}\text{(A) } P\bar{t} = 0, p\bar{t} = 0, f\bar{t} = 0;\\ \text{(B) } t\bar{P}\bar{p}\bar{f} = 0\end{array}\right\} \text{as before,}$$

but (C) $P\bar{R} = 0, p\bar{R} = 0, f\bar{R} = 0,$

and if we further bear in mind that $pf = 0$, then we easily obtain that

$$Rt[P(p\bar{f}+\bar{p}f+\bar{p}\bar{f})+\bar{P}(p\bar{f}+\bar{p}f)]+(R+\bar{R}).\bar{t}\bar{P}\bar{p}\bar{f} = 1. \quad (3.74)$$

The real thus may or may not be in time. But *time is real*, and consists either of the past, the present, or the future; and although there is complete discontinuity between the past and the future, yet there is complete continuity between the past and the present, as also between the present and the future.*

Discontinuity may be nothing more than a contracted or concentrated form of continuity. For instance, within the boundary surface of a liquid there may be continuity: but at the surface there is a continuous transition; rapidly changing near the surface, and apparently undergoing an abrupt change beyond the surface, so that it would appear that a complete discontinuity existed between the two sides of the boundary. Yet the change from one to the other though apparently discontinuous may be regarded as one of rapid or steeply continuous change.

The continuous and the discontinuous therefore, though *apparently* constituting two distinct classes of reality, are found to be reconcilable as one, since the discontinuous may be regarded in the limit as actually continuous. The unity produced through the higher synthesis by the absorption or inclusion of one in the other gives the contradicting elements a consistency otherwise by no means manifest.

As a further illustration of the general method, let us consider the following:

Let R = the real; c = continuous things; d = discontinuous things.

Then in general we must have, as embracing all possible cases in the 'Universe of Discourse',

$$(R+\bar{R}).(c+\bar{c}).(d+\bar{d}) = 1.$$

But since 'Everything is either continuous or discontinuous' and 'Nothing can be both continuous and discontinuous'; and, as we have just shown, 'The discontinuous may in the limit be regarded

* This is the so-called Dedekind cut, according to which all events are either *before* or *after* the present, which separates the two. All points of a straight line fall into two classes, to the right or to the left of one, and only one point, which produces the division of all points into two classes. But the point itself is a third class with a single member. Dedekind regards it as belonging to one or the other. By continuity it would belong to both, and by discontinuity to neither.

as continuous', as a contradictory is also true, then we have the three following equations:

$$(1)\ \bar{c}.\bar{d} = 0;\ (2)\ c.d = 0;\ (3)\ \bar{c}.d = 0.$$

Therefore $$(R+\bar{R}).c.\bar{d} = 1; \qquad (3{\cdot}75)$$

i.e. 'Everything is continuous, and nothing is discontinuous, whether real or not'. Between the real and unreal there is continuity; or else there is no such thing as the 'unreal' and everything is real, that is, if $\bar{R} = 0, \quad R.c.\bar{d} = 1.$

If, however, the unreal is not nothing and may be taken into account, then it seems possible to return to the Hegelian form of the equation and express the matter thus:

$$(R^2+R.\bar{R}+\bar{R}^2).c.\bar{d} = 1; \qquad (3{\cdot}76)$$

since $R^2 \neq R$, because $R.\bar{R} \neq 0$.

This is not a possible conclusion, since the real cannot at the same time be unreal. Hence there can be no such thing as the unreal at all; and everything is real as we had already anticipated.

This leads to the result that on the whole gets over the difficulties just encountered.

Before going further into this highly abstract question let us for the moment consider more fully the purely formal logic as it may be employed in this connexion. These illustrations have been given to show how the method may be applied to the most abstruse and complicated problems. If a question can be discussed at all, it can and should be discussed in this manner.

The continuity we have been considering ought to lead to what Kurt Lewin calls *Genidentität* in his remarkable work entitled *Der Begriff der Genese in Physik, Biologie und Entwicklungsgeschichte* (Berlin: Julius Springer, 1922).

The identity in time is either a continuous recreation as Bergson supposes, or else a continuous reproduction of the 'same' or 'identical' object or idea in time, as a never-ending progressive series. Like a straight line that is in fact curved, the 'identically the same' object or idea becomes altered whether in time or in the logical process of repetition: $x = x^n$ cannot strictly be represented as a linear identity, as Boole supposed; but should be regarded rather as somewhat of the nature of a logarithmic or hyperbolic function such as we have considered on p. 145. The transformations there contemplated have therefore their basis on other grounds.

The idea which is identical with itself admits of an identity in an infinite number of ways, $x = x^n$, where $n = \infty$.

The point is that if an identity is such that when repeated it

undergoes a change not only in a 'space-time continuum', but in what might be called the 'thought continuum'; then x becomes not x; let us say y; and later not y, say z, and so on *ad infinitum*. Hence in the world of 'events' we have also a line of continuous variation in the 'space-time continuum'; a particular case, the curvature of space being a result of the laws of thought.

These variations of a continuous nature may, as we have seen, be due, physically at least, to differences in the rate of change of dimension. Moreover, when they appear as continuous in one scale of being they would be actually abrupt mutations in a higher scale. Continuous variations in the atomic world would be discontinuous mutations in the world of organisms; and in turn continuity in the latter on a microscopic scale could mean discontinuity or abrupt variation in the full-grown and mature organism, such as highly developed animals and plants.

As already shown, if we consider the identity $x^3 = x$, or $x(x-1)(x+1) = 0$, and use the value $x = -1$, the identity will be satisfied, but the relationship of non-contradiction $x(x-1) = 0$ will not be satisfied. If, however, we transform x into $-x$, and put $y = -x$, then $y = 1$, and $y(y-1) = 0$.

That is to say, that for every class of phenomena in which the contradiction of continuity exists there corresponds what might be defined as the negative of the class, for which the law of contradiction does hold good, i.e. discontinuity; and this may therefore be regarded as the noumenon corresponding to the phenomenon; at any rate, so far as the identity $x^3 = x$ is concerned.

In the case of $x^n = x$ there are at least $(n-2)$ possible transformations which will give $y = f(x)$, such that $y(y-1) = 0$ and $y^n = y$. Thus each one of the $n-2$ imaginary roots of the equation

$$(x^{n-2}+x^{n-3}+x^{n-4}+\ldots+x^2+x+1) = 0$$

admits of being transformed, each in its own way into the class that corresponds to the actual reality which does not contradict itself; but which is the truth itself devoid of all appearances or delusions, as the veritable noumenon that stands for these phenomena in all their various aspects. If $n = \infty$, then $n-2$ is infinite, actually so, and the number of imaginary solutions will likewise be infinite. Here then the identity may form a difference through which the class can be transformed and linked to something else. In this way and in this way alone can the unity exist as a plurality. As Kant puts it, 'if the understanding is too small for one conception, it is too big for the other. Neither can be made to fit into its limited and inextensible compass.'

The relationship of subject and object is an analogous case in point. Either can be developed in an apparently endless series of relationships, of subject and object. For, as we have said, the object must itself be the subject of some other idea; and so on apparently *ad infinitum*. Likewise with the subject; it is the object of some other subject, and so on, *ad infinitum*. But as it has already been shown, it is conceivable that subject and object should ultimately become one. If the relationship is an identity the two ends of the series must finally meet.

The Hegelian relationship between opposites is rendered possible only by such differences in identity.

§ 2

The orders of infinity are of special interest in this connexion. A point has been regarded as a unit of infinitesimally small magnitude, yet the line is infinite and made up of an infinite number of points, any two of which have an infinite number of points between them.

Again we may regard the line as a unit, and a plane as an infinite collection of lines. Hence the plane is said to be doubly infinite; being infinite first in regard to the line and then this in regard to the point. Similarly the solid is trebly infinite in regard to the point, and so on. If we take time into consideration then a fourth order of infinity is possible. A line which is infinitely unlimited as regards length (quantitatively) is nevertheless an infinitesimally small unit of a plane, and as such is (qualitatively) limited. 'The infinitely infinite', as Spinoza regarded it, is then the true relation between the various grades of genera and species. Each genus is infinite in relation to the species which it denotes as its predicates, both in space and time. Likewise if it logically connotes any idea as a predicate, the relationship of infinite genus to infinitesimal species holds in either case. Every class contains an infinite number of possible predicates, so that the contraries are included, as in the Hegelian Dialectic. Higher and higher orders of infinity become conceivable; and the 'Infinitely Infinite' is God Himself, the God of Spinoza, in whom obviously consciousness is as essential as the predicate of existence itself. But this would be sheer pantheism were the separate nature of the infinitesimally small as a unit not to be so recognized. The unity of the infinitely great and the infinitesimally small constitutes the real unity; Incarnation, as distinct from Immanence; for the two are totally distinct. The various orders of infinitesimals, whether quantitatively or qualitatively in the Gaussian system of subdivision, lead to different

orders of being. And obviously where the infinitely infinite is considered, the infinitesimally small must have the same significance and importance as the infinitely large, so that magnitude ceases to have value, and universality is the real factor of transcendent moment. We shall see, however, that the number of points in a line is the same as the number in space.*

§ 3

With reference to Peano's work Russell remarks after Peano that 'the whole of Arithmetic and Algebra has been shown to require *three* indefinable notions and *five* indemonstrable propositions. But without symbolism it would have been hard to find this out.'

The three indefinable notions are *zero*, *number*, and *next after*, or 'successor'. According to Russell, these three can be reduced to two notions of *relation* and *class*. Leibnitz, as has been pointed out, endeavoured to reduce all propositions to a few primitive ones by his 'Universal Characteristic'. Boole developed the conception of the characteristic function in the manner already pursued.

I propose to prove in the following manner that if α is a number of the class N_0, then $\alpha+1$ is a number of the class N_0.

For if
$$\alpha = vN_0,$$
$$\alpha^2 = vN_0,$$
$$\alpha^2+\alpha = vN_0, \qquad (3{\cdot}77)$$

that is both classes α^2 and α are numbers
$$\alpha(\alpha+1) = vN_0,$$
$$\alpha N_0(\alpha+1) = vN_0.$$

Therefore if α is N_0, $\alpha+1$ is N_0. The $+$ sign here is for *vel* and therefore includes the alternatives α and 1 in the class $\alpha+1$, so that $+$ means both classes, together or separately. Assuming the distribution law
$$\alpha(\beta+\delta) = \alpha\beta+\alpha\delta,$$
$$\therefore \alpha_{m+1} = \alpha_m+1. \qquad (3{\cdot}78)$$

Only in the case of infinity is $\alpha_{m+1} = \alpha_m$ or $\alpha_m+1 = \alpha_m$. This in fact is Cantor's definition of an infinite number. It will be observed that we have been dealing with classes all the time.

Russell, after Cantor, defines the infinite as follows (*Mysticism and Logic*, p. 86):

* An excellent popular exposition of the infinite is given in a recent book entitled *Number, the Language of Science*, by Thomas Danzig (Allen & Unwin). This might be read in conjunction with Russell's *Introduction to Mathematical Philosophy* (Allen & Unwin).

A collection of terms is infinite when it contains as parts other collections which have just as many terms as it has. If you take away some of the terms of a collection, without diminishing the number of terms, then there are an infinite number of terms in the collection.

Thus the number of even numbers and of all cardinal numbers is the 'same'. For example, in the two rows

$$\left.\begin{matrix} 1, 2, 3, 4, 5, 6, \ldots \textit{ad infinitum}, \\ 2, 4, 6, 8, 10, 12, \ldots \textit{ad infinitum}, \end{matrix}\right\} \qquad (3.79)$$

the number of even must be that of odd and even. The definition of infinity thus avoids a contradiction.

Similarly the number of inductive numbers of the form n^n or n^{n^n} which grow rare as n increases is just the same as that of the inductive numbers.

The name given to this, the smallest infinite class, is the Hebrew *Aleph* $\aleph_0$.

But $2^n > n$ when $n = \aleph_0$. For 2^n is the number of sub-classes of n things. [The proof is given in Russell's *Introduction to Mathematical Philosophy*, p. 85.] He infers that $2^{\aleph_0}$ the number of real numbers is greater than $\aleph_0$. This seems to be contrary to common sense for $2^{\aleph_0}$ is itself a number. But we must adhere to the definition, and it can be shown that the number of sub-classes must be greater than the number of numbers, which might seem to discredit the proof. There is not a one-one relation between them, they are at least equal, and it can be proved that they are not, and must therefore be greater. The alternative may not apply to the infinite.

The number of ratios or fractions is equal to $\aleph_0$, for if we arrange them in a progressive series

$$\left.1 \left(\frac{1}{2}\frac{2}{1}\right) \left(\frac{1}{3}\frac{2}{2}\frac{3}{1}\right) \left(\frac{1}{4}\frac{2}{3}\frac{3}{2}\frac{4}{1}\right) \left(\frac{1}{5}\frac{2}{4}\frac{3}{3}\frac{4}{2}\frac{5}{1}\right)\right\} \qquad (3.80)$$

etc. *ad infinitum*, all ratios occur sooner or later, and according to the definition their number is $\aleph_0$.

But now we have seen that infinity and unity are opposites and that identity is as difficult to define as infinity. In fact, the one is as obscure as the whole. The problem of continuity depends upon identity and difference. And although with Cantor a point, an instant, or an event is something definite and distinct from any other point, instant, or event into which it may change, in actual experience none of these can be conceived. The point or the instant is as much an abstraction as the infinite. The converse of the infinite, the point is infinitely small and such that if infinitesimally small quantities be added to or subtracted from it, it remains

unaltered. If π be the magnitude of a point, then $\pi+\delta\pi = \pi$, and $1+\delta\pi = 1$.

Continuity, then, is nothing more than the infinite applied to the infinitely small in a definite order. The contiguous points merge into one, and yet remain distinct. But the mergence is in the infinite.

Conversely the emergence from unity would depend upon the manner we started, just as in the series (3.83) if we started from the infinite backwards in a particular way we should get back along the inductive numbers or the even numbers. From unity diversity may spring, regarding identity as the infinitely small. One represents the cardinal and the other all real numbers.

Similarly moving upwards from one, a point, we may get two or a multiplicity of units, so that $1 = 2 = n$. This is akin to the phenomenon of reproduction in biology and is possible only if the number of monads is infinite. Here we have a law of emergence, and unity becomes diversity. We may get two when there was only one.

But the number of fractions or ratios is obviously the square of the number of inductive numbers and therefore $\aleph_0^2 = \aleph_0$, and from induction

$$\aleph_0^{n+1} = \aleph_0^n = \aleph_0, \qquad (3.81)$$

like the law of identity.

And from the definition

$$\aleph_0+1 = \aleph_0, \qquad \aleph_0+n = \aleph_0.$$

If we put $x = \aleph_0$ for simplicity:

$$x = x^2 = \ldots = x^n, \quad x+1 = x \quad x+1 = x^2 \quad x^2+1 = x.$$

The first holds also for $x = 0$ or $x = 1$, and all four for $x = \aleph_0$. Therefore

$$x(1-x^2) = 1 \quad x(1-x^2) = -1 \quad \therefore x(1-x) = 0.$$

That is x is not not-x and x that is not x is ± 1.

Strictly speaking $1+x \neq x+1$ when $x = \infty$, and algebra apparently breaks down.

The law of contradiction holds and yet it does not hold. In fact inductive cardinal arithmetic no longer holds when $x = \aleph_0$. There is, however, a particular kind of arithmetic for the infinities. An arithmetic of powers of the infinite or transfinite numbers worked out by Cantor and others. This need not concern us at the moment.*

* $\aleph_0$ includes all cardinal and also all rational numbers. $2\aleph_0$ all real numbers, including transcendental incommensurables such as ϵ and π, but not imaginary or complex numbers like $a+ib$, nor transfinite or numbers of infinite groups.

It may appear that this is pushing things too far. It must, however, be borne in mind that it is a matter of definition. The roots of $x+1 = x^2$ and $x^2+1 = x$, or

$$x^2-x-1 = 0, \text{ and } x^2-x+1 = 0$$

are $$\frac{1}{2} \pm \frac{\sqrt{5}}{2} \quad \text{and} \quad \frac{1}{2} \pm \frac{\sqrt{-3}}{2}.$$

But if $x = x^2$, then $1 = 0$; which means that in dealing with infinite numbers 1 does not count, as in fact the definition implies. When the number of numbers in two classes is infinite, the classes merge into one. If the number of monads in the Universe is infinite they would merge into one, with all due respect for Dr. McTaggart. It would be absurd to say that the infinite is reality, or that it exists or that it is perfect. It is quite a different order of being from the finite, and entirely transcendental. In it alone do subject and object become one as a class. Antecedence, like succession, has no meaning with regard to it, and causation loses its significance. It is and is not its own cause, or rather has no cause. Like freedom, chance becomes purpose and purpose chance.

It is no longer right to say that a point is one. It is the infinitely small and as such akin to the infinite itself $= \frac{1}{\aleph_0} = \aleph_0^{-1} = \aleph_0$.

Any number has been expanded in a series of fractions *ad infinitum* thus

$$n = \frac{1}{m} = \frac{1}{(m+1)}\left(1+\frac{1}{m}\right) \tag{3.82}$$

$$= \frac{1}{(m+1)} + \frac{1}{(m+1)^2} + \frac{1}{(m+1)^3} + \ldots + \frac{1}{(m+1)^p}\left(1+\frac{1}{m}\right).$$

If $p = \infty$ the series becomes a progression *ad infinitum*.

As a particular case $$1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{\aleph_0} + \ldots + \frac{1}{2^{\aleph_0}}. \tag{3.83}$$

We have already seen that $2^{\aleph_0} > \aleph_0$.
So that there is a closer degree of identity in unity than might be imagined by merely writing

$$1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{\infty}.$$

One gives the lower limit, but the other unity itself: so then

$$0 \text{ is not } \frac{1}{\aleph_0} \text{ but } \frac{1}{2^{\aleph_0}}.$$

As Cantor has shown, $2^{\aleph_0}$ is the number of points in space or of instants in time (see Russell, *Introd. Math. Phil.*, p. 86). It is also 'the number of points in any finite portion of space, whether line, area, or volume'. This applies to monads as well as to points. Monads or points like infinite cardinals belong to 'reflective' classes. The Ego belongs to the reflective class: a reflective class being one which is similar to—i.e. has a one-one relation to a proper part of itself, a proper part being a part short of the whole. The subject bristles with difficulties, but modern mathematics has grappled with it in a masterly manner.*

* It would be going beyond the limits of the present work to enter fully into this question, however fascinating and related to the subject; Russell and Fraenkel (loc. cit.) should be referred to in this connexion. My concern is with the logical continuity of classes generally as well as with the identity of points.

CHAPTER XII

THE HEGELIAN DIALLAGE

THE Hegelian dialectic is a diallage consisting of a process somewhat as the following. We may say in the first instance that $X = x$; and yet that this is not strictly the case. If it is strictly true, then $X = x\bar{y}$. But if not so, then $X = \bar{x}y$ suppose; and if this is not actually true, then $\bar{x}\bar{y}$. But

$$xy+(x\bar{y}+\bar{x}y+\bar{x}\bar{y}) = 1.$$

Therefore if $\bar{x}\bar{y}$ is not true either, must xy be so. Now let us take (xy) as a group together and put $xy = p$. The process may be repeated by similar negations.

X is $p\bar{z}$, negating this it is $\bar{p}z$; then on further negation it is neither p nor z, $\bar{p}\bar{z}$; which not being true either, we are driven back to pz. So that $X = pz = xyz$, since $(pz)+(p\bar{z}+\bar{p}z+\bar{p}\bar{z}) = 1$. In this way we arrive at the result that $X = xyzw\ldots$. Instead of three negations Hegel performs only two: first from $x\bar{y}$ to $\bar{x}$; second from $\bar{x}$ to xy; which is obviously the same thing; for it is necessary to negate y before we can return to xy. The negation of $\bar{x}y$ is not xy, but $\bar{x}\bar{y}$; but the negation of the latter following the previous ones is then xy. He may mean, however, to go from $x\bar{y}$ to $\bar{x}(y+\bar{y})$ and thence to xy, which would be correct.

If we were to proceed from $x\bar{y}$ to $\bar{x}y$, and thence direct to xy, this could be negated to $\bar{x}\bar{y}$, as the synthesis of the process; equivalent to $\bar{x}\bar{y}z$; which if negated, $(x\bar{y}+\bar{x}y+xy)$ being already negated, becomes $\bar{x}\bar{y}\bar{z}$. The final synthesis would then be $\bar{x}\bar{y}\bar{z}\bar{w}\ldots$; or absolute negation. This is the very opposite of $xyzw\ldots$.

As a matter of fact the combined process is the proper procedure, so that we should arrive at the result

$$xyzw\ldots+\bar{x}\bar{y}\bar{z}\bar{w}\ldots = 1.$$

This is the totality.

Hence $x = y = z = w\ldots$, as Bradley maintained, which is equivalent to $x = x.x.x.x\ldots$, or $x = x^n$; or, as we have already seen,

$$x.(1-x).(1+x+x^2+x^3+\ldots+x^{n-2}) = 0.$$

The imaginary and negative roots of unity suffice to give the apparent unity in differences which contradict each other; whereas the actual identities form the reality of the Logos or Idea.

In general by elementary algebra we may express $x_0x_1x_2x_3\ldots x_n$, in the form of a series thus:*

$$x_0x_1x_2\ldots x_n = x_0+x_0(x_1-1)+x_0x_1(x_2-1)+ \qquad (3.84)$$
$$+x_0x_1x_2(x_3-1)+\ldots+x_0x_1x_2x_3x_4\ldots x_{n-1}(x_n-1).$$

* This is one form of Zermlo's Theorem.

Hence,

$$1-(x_0x_1x_2x_3\ldots x_n) = (1-x_0)+x_0(1-x_1)+x_0x_1(1-x_2)+\ldots+ \\ +x_0x_1x_2x_3\ldots x_{n-1}(1-x_n), \quad (3.85)$$

and again

$$x_0 = x_0(1-x_1)+x_0x_1(1-x_2)+x_0x_1x_2(1-x_3)\ldots+x_0x_1x_2x_3\ldots x_\infty = x_0,$$

or otherwise

$$x_0 = x_0x_1x_2x_3\ldots x_\infty+x_0x_1x_2\ldots x_{n-1}(1-x_n)+\ldots+ \\ +x_0x_1x_2(1-x_3)+x_0x_1(1-x_2)+x_0(1-x_1) = x_0. \quad (3.86)$$

In the simple case of four classes x, y, z, w, we have the obvious classification

$$x = x\bar{y}+xy\bar{z}+xyz\bar{w}+xyzw;$$

or, what is the same thing with the order reversed,

$$x = xyzw+xyz\bar{w}+xy\bar{z}+x\bar{y}. \quad (3.87)$$

Either of these two latter is the ordinary logical method of classification.

Hence if $x = xyzw$ it is not necessary that $x\bar{y} = 0$, $y\bar{z} = 0$, $z\bar{w} = 0$; it is sufficient that $x\bar{y} = 0$, $xy\bar{z} = 0$, $xyz\bar{w} = 0$, that is, that every x is y and every y that is x is z; and every z that is y and x is w. In other words, it is the identical x that runs through the series; and it is with that we have to deal in the series of premisses. This is the sort of argument made use of in the ontological proofs. The central idea is the essential one common to all the classes.

In the case of the Ego x which includes the manifold

$$x_1x_2x_3x_4\ldots x_n,$$

if x_2 is the class of existent things, and x_1 self-conscious ones, then $x_1(1-x_2) = 0$, and $x_0(1-x_1) = 0$; so that x_0, x_1, x_2 occur throughout. This may appear to be a mere syllogistic argument, which in fact it is, but it shows how and why the essential elements remain throughout as factors in the series, however much the members differed in all other respects. The Ego, self-consciousness, and existence are inseparably one.

We have already seen that, in general,

$$x_0 = x_0(1-x_1)+x_0x_1(1-x_2)+x_0x_1x_2(1-x_3)+\ldots+ \\ +x_0x_1x_2x_3\ldots x_{n-1}(1-x_n)+x_0x_1x_2\ldots x_n = x_0;$$

$$\therefore \quad x_0 = F(x_0, x_1, x_2, \ldots, x_n) \\ = \sum_0^n x_0x_1x_2\ldots x_n\bar{x}_n+x_0x_1x_2\ldots x_n = x_0. \quad (3.88)$$

If each of the terms vanishes, except the last, so that

$$\begin{aligned} x_0(1-x_1) &= 0, \\ x_0x_1(1-x_2) &= 0, \\ x_0x_1x_2(1-x_3) &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ x_0x_1x_2\ldots x_{n-1}(1-x_n) &= 0, \end{aligned} \tag{3.89}$$

then, $$x_0 = x_1 = x_2 = \ldots = x_n = x_0.$$

From (3.88) $\bar{x}_0 = \bar{x}_0x_1+\bar{x}_0\bar{x}_1x_2+\ldots+\bar{x}_0\bar{x}_1\bar{x}_2\ldots\bar{x}_n.$

Hence

$$1 = x_0+\bar{x}_0 = (x_0\bar{x}_1+\bar{x}_0x_1)+(x_0x_1\bar{x}_2+\bar{x}_0\bar{x}_1x_2)+\ldots \\ +(x_0x_1x_2\ldots x_n+\bar{x}_0\bar{x}_1\bar{x}_2\ldots\bar{x}_n) = 1. \tag{3.90}$$

Each of the terms within brackets consists of a particular aspect of x and its *opposite*; so that when all such are negated, we are left with the whole truth and its opposite; ultimately in fact when $n = \infty$; with the Absolute and its negation, to form the totality.

From (3.85) it follows that since

$$1-(x_0x_1x_2x_3\ldots x_n) = (1-x_0)+x_0(1-x_1) \\ +x_0x_1(1-x_2)+\ldots+x_0x_1x_2x_3\ldots x_{n-1}(1-x_n),$$

$$1-(\bar{x}_0\bar{x}_1\bar{x}_2\bar{x}_3\ldots\bar{x}_n) = (x_0)+\bar{x}_0x_1+\bar{x}_0\bar{x}_1x_2+\ldots+\bar{x}_0\bar{x}_1\bar{x}_2\bar{x}_3\ldots\bar{x}_{n-1}x_n, \\ = (x_0)+(x_1)+(x_2)+(x_3)+\ldots+(x_n); \tag{3.91}$$

where each term (x_1), (x_2) ..., (x_n) is distinct from the preceding ones.

If we take the particular case of two classes x and y, we have

$$1-xy = \bar{x}+x\bar{y}, \quad \text{and} \quad 1-\bar{x}\bar{y} = x+\bar{x}y.$$

Similarly

$$1-xyz = \bar{x}+x\bar{y}+xy\bar{z}, \quad \text{and} \quad 1-\bar{x}\bar{y}\bar{z} = x+\bar{x}y+\bar{x}\bar{y}z.$$

Moreover, since in the algebra of logic the numerical coefficients do not enter into account, we may also write

$$1-xy = \bar{x}+\bar{y}+x\bar{y}+\bar{x}y, \quad \text{and} \quad 1-\bar{x}\bar{y} = x+y+x\bar{y}+\bar{x}y,$$

and likewise,

$$1-xyz = \bar{x}+\bar{y}+\bar{z}+(x\bar{y}+y\bar{z}+z\bar{x})+(xy\bar{z}+yz\bar{x}+zx\bar{y});$$

so that, similarly,

$$1-\bar{x}\bar{y}\bar{z} = x+y+z+(\bar{x}y+\bar{y}z+\bar{z}x)+(x\bar{y}\bar{z}+y\bar{z}\bar{x}+z\bar{x}\bar{y}), \tag{3.92}$$

the last terms in brackets being $(x)+(y)+(z)$; where (x), (y), (z) represent the x's, y's, and z's that are absolutely independent of each other.

It thus appears that (3.91) may be written

$$1-(\bar{x}_0\bar{x}_1\bar{x}_2\bar{x}_3\ldots\bar{x}_n) = \sum x_0+\sum\bar{x}_0x_1+\sum\bar{x}_0\bar{x}_1x_2+\ldots \\ +(x_0)+(x_1)+\ldots+(x_n). \quad (3.93)$$

The separate and distinct elements are connected in no way with each other, but as a group form the negation of all else. These correspond to the natural numbers in arithmetic; the overlapping Hegelian classes and the irrational and incommeasurables to the others. The theory of Type appears to overlook them.

The negation of the negation of the Absolute $x_0x_1x_2x_3\ldots x_n$ includes all these as well as the Absolute, and the series of terms that overlap or have something in common.

It cannot be said therefore that the unity of all independent and distinct elements, such as *selves*, is itself a conscious unit, and it is not the Absolute; but the Absolute is none the less a unit.

The unity embracing all the elements combined is the truth or Absolute.

This appears to have been the conclusion arrived at by Bradley; and it is obvious from the results thus obtained that the difference between him and McTaggart is easily accounted for when the problem is thus analysed.

That which is absolutely distinct from everything else can have nothing in common with anything else. But selves which are conscious have at least consciousness in common, which though in each Ego is differentiated, none the less forms the common element uniting them together. All these Egos combined do not form a single Ego, which was McTaggart's point; but the consciousness of the Absolute is not denied on that account, which was Bradley's point. The confusion that has arisen from the identification of the unity of the Absolute with the ideal unity of all possible selves is easily avoided by thus analysing the problem.

The consciousness of the Absolute Idea that we as individual monads share with the Absolute Monad is not the Absolute Idea or Monad, though we may be conscious of it. If we are all thus united in One it is not in one personality, but in sharing a common consciousness or Spirit in part which is the essential element of each monad, and particularly so of the Absolute Monad which embraces all.

CHAPTER XIII

THE PROBABLE AND APPROXIMATELY TRUE SYLLOGISM

§ 1

WE now turn to another aspect of the question.

In the Hegelian dialectic we have obviously various degrees of truth and all things are possible, though not all things are probable or expedient. We may in fact employ the calculus of probability in the place of the Hegelian logic. For an approximation to the truth must be expressible like a probability.

The most important application of this theory of probability to the logic of practical questions such as we have been discussing—for they are strictly speaking practical—is in the estimation of probabilities in a hypothetical syllogism. This in its naked form we may express thus:

Major premiss: If the proposition Y is true, X is true.

Minor premiss: If the proposition Z is true, Y is true.

Conclusion: If the proposition Z is true, X is true.

We may put p and q for the probabilities of the major and minor premisses respectively; the probability of the conclusion r can be calculated by the calculus of logic (see Boole's *Laws of Thought*, pp. 300, 301) $r = pq + a(1-q)$. This will, no doubt, seem obvious on reflection, but Boole gives a formal proof, where a is an arbitrary constant equal to the probability that if the proposition Z is true and Y not true, then X is true—namely, the probability that if the minor premiss is false, the conclusion is true.

If, as a particular case Z is true, then the Prob. $y = q$; and the Prob. $xy = pq$; so that the Prob. $x = pq + a.(1-q)$.

This is what common sense requires; for the probability of x is the probability that both premisses are true, together with the probability that if the minor premiss is false, the conclusion is nevertheless true. The latter is, however, usually overlooked and often the probability of the conclusion taken erroneously for the product of that of the premisses.

The general method of analysis given by Boole will repay perusal notwithstanding the severe analytical labour involved in cases which often admit of solution by simpler means. The point about his method, however, is that *it is general* and admits of application to all problems of probability in a strictly logical manner. The mastery of the method is largely a matter of practice.

We have considered the application of the principle of induction to classes involving connotation as well as denotation.

From the calculus of probability a new kind of logic can be evolved; the nature of the inferences of which would be quite different from those of ordinary formal logic. For if we consider any two propositions whatever, not necessarily logically related, whose probabilities are p and q respectively; the probability r of a third is given by the expression

$$r = c.pq + a.\bar{q}; * \qquad (3.94)$$

where c is a constant expressing the probability that if the two propositions be conjointly true the third is also true. When the third follows logically and necessarily from the other two as premisses $c = 1$; but not otherwise. c is the probability that the inference is correct. If the argument is a *non sequitur* $c = 0$, and $r = a.\bar{q}$; which is the probability that if the minor premiss is false the conclusion is nevertheless true.

Since in the syllogism $r = pq + k\bar{q}$, where $k < 1$, then

(1) If $a = r - p$, and $b = r - q$, the increase in the probability of the conclusion over that of the premisses separately is for the major $a = pq - p + k\bar{q} = k\bar{q} - p\bar{q} = \bar{q}(k - p)$. (1.1)

Hence the probability of the conclusion is greater than that of the major premiss only if it is *a priori* more likely than the latter; but not otherwise. And for the minor

$$b = r - q = pq - q + k\bar{q} = q(p - k) + (k - q) = k\bar{q} - \bar{p}q. \qquad (1.2)$$

Here if the *a priori* probability of the conclusion is intermediate between those of the premisses, the probability of the conclusion from the premisses is greater than that of the minor premiss and less than the major. In other words, if $p > k > q$; then $p > r > q$.

* If the probabilities of the premisses be reversed, so that q is that of the major and p that of the minor, we have:

$$r_{qp} = qp + a_2\bar{p} \quad \text{and} \quad r_{pq} = pq + a_1\bar{q},$$
$$\therefore \quad r_{qp} - r_{pq} = a_2\bar{p} - a_1\bar{q}. \qquad (3.95)$$

Hence if we consider two propositions, $\phi = 0$, $\psi = 0$, whose probabilities are p and q respectively, the resultant or conclusion $\chi = 0$ whose probability is r_1 will depend upon the order, as in the case of quantum mechanics. Since ϕ and ψ may vary as waves of probability:

putting
$$p = \frac{e^x + e^{-x}}{2}, \quad q = \frac{e^y + e^{-y}}{2},$$
$$r_{pq} = \cos ix \cos iy - a_1 \sin \frac{iy}{2}, \qquad (3.96)$$

and
$$r_{pq} - r_{qp} = a_2 \sin \frac{ix}{2} - a_1 \sin \frac{iy}{2},$$

if ϕ and ψ represent states $\phi\psi - \psi\phi \neq 0$ as in quantum mechanics. (3.97)

It also follows from (1.2) that $k\bar{q}-q\bar{p} > 0$; or $\frac{k}{q} > \frac{\bar{p}}{\bar{q}}$.

So that the conclusion is more probable than the minor premiss if the ratio of its *a priori* probability to that of the minor is greater than that of the negation of the major to that of the negation of the minor.

(2) If $p = q$ then $a = b$. Therefore if the two premisses are equally likely, the conclusion is more, or less, likely than either premiss, according as it is *a priori* more, or less, likely than either premiss.

(3) If $p = 1$, then $a = \bar{q}(k-1)$, and $b = q(1-k)+k-q = k\bar{q}$. a is therefore negative and b positive; and the conclusion is always more probable than the minor premiss, whatever that might be, and less probable than the major.

(4) If $q = 1$, then $a = 0$, and $b = -\bar{p}$, and therefore negative. Hence if the minor is a certainty, but the major only probable, the conclusion cannot be a certainty—as common sense requires—but equals the probability of the major.

(5) But if $p = 1$, and $q = 1$, then $r = 1$, i.e. from two certain premisses the logical conclusion is likewise a certainty.

(6) If $k > p$ and $\frac{k-q}{k-p} > q$ the conclusion is more probable than either premiss. Hence a higher probability than either premiss is possible only by the aid of *a priori* knowledge.

(7) $r-p = \bar{q}k-\bar{q}p$, $r-q = \bar{q}k-q\bar{p}$, so that the conditions that the conclusion should be more probable than the premisses are

(*a*) $\bar{q}k > \bar{q}p$ or $k > p$ and (*b*) $\bar{q}k > q\bar{p}$ or $k > \frac{\bar{p}}{\bar{q}}.q$.

§ 2

We may now calculate the probability of a conclusion in a long series of premisses as in a sorites. Thus

If Y is true, X is true; Prob. p. (1)
If Z is true, Y is true; Prob. q. (2)
If W is true, Z is true; Prob. r. (3)
If S is true, W is true; Prob. s. (4)

Then

If Z is true, X is true $= pq+c_1.\bar{q}$. (2′)
If W is true, X is true $= r.(pq+c_1\bar{q})+c_2.\bar{r}$, (3′)
$= r.q.p+c_1.r.\bar{q}+c_2.\bar{r}$. (3.98)
If S is true, X is true $= s.(rqp+c_1r\bar{q}+c_2\bar{r})+c_3\bar{s}$. (4′)

The constants c_1, c_2, c_3 are the probabilities that X is true even if the premisses (2), (3), (4) are false.

Now when $c_1 > p$ it follows that (2′) is greater than p, that is, the probability is increased. Similarly, if $c_2 > pq+c_1\bar{q}$, or if $c_3 > rqp+c_1r\bar{q}+c_2\bar{r}$, the probability goes on increasing. We see, then, how it is possible that the probability of a conclusion can increase with the chain of reasoning. Moreover, it is clear that the probability of X depends upon that of its truth in the event of each of the premisses, other than the major premiss in which it occurs, being false; the probabilities c_1, c_2, c_3, entering into the calculation of the resultant probability. These correspond to what Newman would perhaps have called the probabilities of the 'illative sense'. They are of particular interest and importance in the estimation of probabilities in moral problems; and particularly theological questions in which the personal element is of considerable importance. In this respect then we may take the example (example § 6) relating to the probability of miracles recorded in the Bible. The *a priori* element in judgement plays a most important part in all rational processes; and whether we regard it as an intuition or a prejudice, unconscious bent or predilection, it constitutes an indispensable part of every probable inference, as distinct from the ordinary logical syllogism. This is the *illative sense*; a very dangerous faculty, as dangerous as unlawful reason itself, to the credulous or the biased and the unscrupulous. Yet no sane judgement seems possible without it in dealing with the belief in Nature, and still more with the affairs of men. When we say that a man has sound judgement, we mean that he exercises this faculty with discretion, moderation, caution, and good sense. But his innate and acquired propensities enter with relentless effect into the formation of his convictions the more conscientiously and rigorously he may endeavour to arrive at the correct results. For his judgement finally is the outcome of his own preconceived ideals and disposition, far beyond his own control or conscious apprehension on the formation of judgement by apparently impartial deliberations. As his comprehension is limited by the horizon of his knowledge and his intellectual vision, so is his judgement bounded and constrained by the very limitations of his character, the unconscious assertions and mute reservations of his nature as an intelligent but human being.

The distinction between *notional* and *real* assent must here be emphasized. Notional assent is the assent given to an abstract proposition, a truth derived solely by the process of rational inference, without regard to its *a priori* probability in the language

of our scientific calculus. But a real assent is the outcome of one's own personal experience given to a particular and concrete fact, and embodies as it were, not the intellect alone, but the very heart and soul as the essential features of the mind by which and through which it is expressed as a living reality, a living truth, which serves to animate our whole being, our outlook and our conduct, with a permanent influence as a guiding force; a conation as distinct from a cognition, as the power of music, or poetry, or art, or still more religion itself, upon the refined instincts of our nature. Yet, as I say, it is a most dangerous faculty to deal with in the pursuit of truth, unless when duly exercised it is employed, as it should inevitably be, as the handmaid and faithful attendant in the search for truth as a living reality and not as a mere empty formula of scholastic logic. And Bradley realized all this in recognizing feeling as an essential element of logical faculty and every logical proposition.

In the case of a series of hypothetical propositions we can write down the probability of the conclusion in the following manner.

If π_n = prob. of conclusion from $(n+1)$ premisses, whose respective probabilities are $p_1, p_2, \ldots, p_{n+1}$; then

$$\pi_n = p_{n+1}\{p_n(p_{n-1}(p_{n-2}\cdots p_4[p_3(p_1p_2+c_1\bar{p}_2)+c_2\bar{p}_3] +c_3\bar{p}_4+\ldots+c_n\bar{p}_{n+1}))\}. \quad (3.99)$$

In other words,

$$\pi_n = p_{n+1}p_np_{n-1}\cdots p_2p_1+c_1\bar{p}_2p_3\cdots p_{n+1}+c_2\bar{p}_3p_4\cdots p_{n+1} +c_3\bar{p}_4p_5\cdots p_{n+1}+\ldots+c_{n-1}\bar{p}_np_{n+1}+c_n\bar{p}_{n+1};$$

c_n being the probability that if the conditioned proposition in the first or major premiss be true, but the last premiss whose probability is p_n be false, the conclusion π_n is nevertheless true.

In general $$\pi_n = p_{n+1}\pi_{n-1}+c_n\bar{p}_{n+1}. \quad (4.00)$$

The determination of the constants $c_1, c_2, c_3, \ldots, c_n$ is of importance in some cases, and many fallacies arise through neglect of these most significant 'factors'. In the case of all inductions obviously the actual value of the probability of the major premiss, taking the probabilities of the minor premiss and the conclusion as empirical facts, depends upon this constant c_n. And in the case of a sorites upon a series of such constants—each of which is the probability that if the predicate or conditional clause in the major premiss is true, but the minor premiss to which it refers is false—the conclusion is nevertheless true. Thus in the example given in § 6 of the probability of the miracles recorded in the Bible, account must be taken of the probability of the miracles independently of that of any one of the premisses.

In the case of a single syllogism, if p_1 and p_2 be the probabilities of the premisses and π_1 that of the conclusion,

$$(1)\ c_1 = \frac{\pi_1 - p_1 p_2}{\bar{p}_2},\quad (2)\ p_1 = \frac{\pi_1 - c_1 \bar{p}_2}{p_2},\quad \text{and } (3)\ p_2 = \frac{\pi_1 - c_1}{p_1 - c_1}. \tag{4.01}$$

It depends upon which are the given facts; assuming that the probability of the 'conclusion' is given empirically, that of either premiss may be found if that of the other is given when the prejudice is known. It must not be supposed that this inverse process is to be identified with induction; in which the inference is from the minor premiss and 'conclusion' to the major premiss. The determination of the constant c_1 is the elusive 'factor' in the proceedings. In most cases it is really indeterminate, or else negligible, or assumed to be such for practical purposes.

Thus let Prob. 'if Y is true, X is true' $= p$;
Prob. 'Y is true' $= a$.

Then as before

Prob. 'X is true' $= a.p + c.\bar{a}$;

i.e. the probability of X is the sum of the probabilities, that Y and X should both be true, and that if Y is not true yet X should be true without it; in other words it is the probability that X should be true whether Y be true or not.

Similarly the prob. if X is true, Y is true $= \dfrac{\text{Prob. } Y.X}{\text{Prob. } X}$.

Prob. If X is true, Y is true $= \dfrac{a.p}{a.p + c.\bar{a}}$.

But Prob. $\bar{X} = 1 - (ap + c\bar{a})$; Prob. $\bar{Y} = \bar{a}$; therefore Prob. $\bar{Y}\bar{X} = \bar{a}\{1 - (ap + c\bar{a})\}$.

The Prob. If X is not true, Y is not true $= \dfrac{\bar{a}\{1 - (ap + c\bar{a})\}}{\{1 - (ap + c\bar{a})\}} = \bar{a}$.

Consequently, if the probability that if Y is true X is true be p, and that Y is true be a; then the probability that if X is not true Y is not true, is not as might be imagined $\bar{p}$, but $\bar{a}$, namely the probability that Y is not true; as a little consideration will show to be correct. It matters not in the very least what the probability of X might be, so far as the probability of the converse hypothetical proposition may be concerned; which obviously depends upon the probability that Y is not true, and upon nothing else.

Similarly the probability that if $\bar{X}$ is true Y is not true is also the probability that Y is not true; for it is $\dfrac{(ap + c\bar{a})\bar{a}}{(ap + c\bar{a})} = \bar{a}$. So that

the probability that Y is not true is independent of whether X is or is not true; as might at first sight have been foreseen.

In the Hegelian dialectic it is not the probability but the approximation to the truth that is implied. It will be observed that in either case we have to deal simultaneously with opposites, and the fractions which represent the truth in one case, and the probability in the other follow similar laws, and may be dealt with in a similar manner. The degree of truth 'das Moment' may be represented like the probability, as a fraction of unity; unity standing for the absolute truth. The distinction between probability and approximate truth, however, cannot be too strongly emphasized; striking as the analogy between them may be. [In either case the algebraic sum of the two alternatives, the proposition and its opposite, is equal to unity.]

There is thus a danger of confusion between an approximation to the truth and its probability; two different things which may be treated in many instances in the same manner.

§ 3

If p be the probability that a witness speaks the truth, and q the probability of a certain event; and if there are m witnesses who are agreed that the event took place: assuming these witnesses to give their evidence independently of each other, and not to have been subject to any influence, such as that of hyper-suggestion or otherwise; so that the credibility of each witness is the same whether taken separately or collectively; then the probability that if they all asserted that the said event took place it actually did so is $= p^m.q$. But the probability that they should agree if the event did not take place is the probability that their statement is not true, multiplied by the probability that the event should not have taken place; $= (1-p)^m.(1-q)$.

Consequently the probability that they should agree about the event whether it actually took place or not $= p^m q+(1-p)^m(1-q)$. And let u equal the probability that if they agreed the event took place, then

$$u=\frac{p^m q}{p^m q+(1-p)^m(1-q)},$$

therefore

$$\left(\frac{p}{\bar{p}}\right)^m\left(\frac{\bar{u}}{u}\right)=\frac{\bar{q}}{q}. \qquad (4.02)$$

Or if a = credibility of each witness, i.e. $\frac{p}{\bar{p}}$, and b = the credibility

of their combined testimony, $\frac{u}{\bar{u}}$,

$$\frac{a^m}{b} = \frac{1-q}{q}, \text{ i.e. } q = \frac{1}{\frac{a^m}{b}+1}.$$

Let us suppose the credibility of each witness is ten to one, i.e. $a = 10$; and that their combined credibility is likewise ten to one; $b = 10$. Then

$$q = \frac{1}{10^{m-1}+1}. \tag{4.03}$$

Therefore if $c = \frac{1-q}{q}$, the *a priori* incredibility of the event, $c = 10^{m-1}$;

therefore

$$m = 1+\log c. \tag{4.04}$$

This gives the number of independent credible witnesses necessary to render an event of an *a priori* incredibility c actually credible as 10 : 1 true. Hume supposed that the incredibility of a miracle would be infinite; and therefore the number of witnesses required would be infinite. But it appears from the above law that if twelve credible witnesses assert a fact to have occurred, its *a priori* incredibility could be as great as $10^{11} : 1$, i.e. one hundred thousand millions to one; and yet its credibility in view of the evidence would be ten to one; and no more incredible than any ordinary assertion by any one of the witnesses about any ordinary phenomenon of everyday life.

If p_1 = probability that a witness A speaks the truth; and p_2 that B speaks the truth; and r is the probability that they agree about a certain event, the *a priori* probability of which is q; then the probability that if they agree the statement is true is $u = \frac{p_1 p_2 q}{r}$.

Now $r = p_1p_2+(1-p_1)(1-p_2) = 2p_1p_2+1-p_1-p_2$.

Therefore

$$u = \frac{q\{p_1+p_2-(1-r)\}}{2r}.$$

And if $p_1 = p_2$,

$$u = \frac{q}{2}.\frac{\{2p-(1-r)\}}{r}. \tag{4.05}$$

Consequently if the probability of each witness's speaking the truth is one-half that of his not agreeing with another, the evidence on the whole has no value.

Similarly, if the average of the probability of two independent witnesses whose credibility differs be half the chance of their differing, the evidence has no value whatsoever.

The probability u_n where there are n witnesses is $u_n = \frac{p^n \cdot q}{r}$.

Therefore, $$p = \left\{\frac{(ru_n)}{q}\right\}^{\frac{1}{n}}. \qquad (4.06)$$

If now the witnesses are sure to agree, $r = 1$; and since p must be less than unity, u_n must be less than q; and if further q is extremely small, u_n is still smaller; so that whatever the credibility of each witness might be, the chance that if they are sure to agree the statement is true must be less than that of the event. Moreover, in general the greater the chance of their agreeing the less the chance that what they say is true, if they agree. On the other hand, for a given event and decision or judgement, the greater the chance of unanimity the greater the credibility of each witness. Further, if the chance of unanimity be normal, and that of the event recorded extremely small, the value of the evidence on the whole is very small, even if the credibility of each witness be normal. The combined evidence therefore only tends to discredit each witness. For it would be equal to evidence for a normal event, when the witnesses are untruthful or prejudiced and likely to agree. This is the general rule in the psychology of crowds.

It is easy to see that if twelve persons of ordinary credibility 10:1 agree about an ordinary event, and the credibility of their judgement is accepted as of normal value 10:1; the chance of their having come to such a unanimous decision is 1/3, $r = \left(\frac{10}{11}\right)^{12}$ or $r = 0{\cdot}3187$, or 2:1 against their having so agreed. But if they are experts, or on oath, their credibility might be taken at 100:1, i.e. the probability of their speaking the truth 100/101. In this case the chance of their agreeing is 9/10. Hence $r = \left(\frac{100}{101}\right)^{12}$ or $r = 0{\cdot}8875$, or approximately 0·9 or 9:1 that a jury would agree. This fully justifies the jury system in England.

§ 4

In general with regard to the calculation of probabilities in a hypothetical syllogism, the conclusion is more likely than either of the premisses if $k > p$ and $k > \frac{\bar{p}}{\bar{q}}q$, or since then $\bar{k} < \bar{p}$ $\frac{k}{\bar{k}} > \frac{q}{\bar{q}}$. And in the case of a sorites the probability would go on increasing to a greater and greater probability. The same considera-

tions apply to the syllogisms in which p and q stand for the degree of approximation to the truth; for clearly the approximation of the conclusion must likewise be greater than that of either premiss; so that by a sorites of that nature, if each premiss contains more truth than falsehood, the conclusions will approximate more and more to the truth.

If we take, however, a proposition p, and its opposite or antithesis $\bar{p}$; then obviously the probability, or approximation to the truth, is greater in a syllogism that is logically correct than in one that is a *non sequitur*, for the probabilities are $pq+k.\bar{q}$, and $c.\bar{p}q+k.\bar{q}$, respectively (provided $p > c\bar{p}$).

c is the probability that if the minor premiss is true but the major not true, the conclusion is true. Clearly if $c = 1$; for p, it is < 1 for $\bar{p}$ as a *non sequitur*.

That is if the syllogism is logically correct, the probability of the conclusion is less probable than if for the major premiss we substitute its contradictory, and draw the same but illogical conclusion; provided $c > \frac{p}{\bar{p}}$; for $r = pq+k\bar{q}$; $r' = c\bar{p}q+r\bar{q}$. Consequently

$$r'-r = (c\bar{p}-p)q, \text{ therefore } c\bar{p} > p. \qquad (4.07)$$

If p stands for the probability that 'all men are immortal'; and q for the probability that 'Socrates was once a man', the probability that he was immortal as drawn logically from the unlikely major premiss that all men are immortal is less than the inference from the more likely major and contradictory 'Some men are not immortal', but Socrates is an exception, provided that the chance that although some men are not immortal yet he is so, is greater than the ratio of the probability that all men are immortal, to that of its contradictory, that some men are not so.

Under such prescribed circumstances therefore, the probability of the truth of an illogical inference from premisses that are probably true may be very much greater than the probability of a perfectly logical conclusion from premisses that are unlikely.

Thus in the Hegelian dialectic if we substitute the contradictory of the major premiss in the hypothetical syllogism we arrive at similar results.

But if we substitute the contradictory of the minor premiss we obtain $c_1.p.\bar{q}+k_1q$ for the probability or approximation of the conclusion, where c_1 is the chance or approximation that the conclusion is true when the major is true but the minor not true; and k_1 is the probability that if the minor premiss is true the conclusion is true.

These constants are constants of the 'illative sense'. They may also be regarded as factors which determine the validity of all processes of the understanding, as in the intuitions of space and time; as well as those of the Categories, in the Kantian philosophy. An infinite series of propositions ever approximating to the truth will never enable us to arrive at the absolute truth unless there is a factor in the illative sense that compensates for the imperfection of the synthetic approximation.

The degree of truth therefore in the negation of the illative sense must be equal to that of the synthesis, if the latter can attain to absolute truth. For then $pq+k\bar{q} = 1$. Even if $k = 1$, $1 > pq+\bar{q}$ because $pq+\bar{q} = pq+1-q = 1-q\bar{p}$.

This is equivalent to the *a priori* synthesis required by Kant. It is the final jump that the Hegelian dialectic does not really afford. And it is the point that Hegel's magnificent literary style serves most eloquently to glide over and conceal. The unity of identity in the Absolute can never be attained through mere dialectic.

The probability or approximation to the truth of the two premisses combined, should be the same as the probability or degree of approximation that the conclusion should be true when the minor premiss is false; that is if $q = 0$ $\bar{q} = 1$. $\therefore$ $r = k$. This margin separates approximate from absolute truth.

We may arrive at a condition that must hold if the conclusion is to be absolutely true, in the following manner.

If the conclusion is absolutely true, $1 = pq+k\bar{q}$. Therefore $pq = 1-k\bar{q}$; that is, the probability that both premisses are true must be the same as that $k\bar{q}$ is false. Yet contrary to apparent common sense this is not the same as the probability that neither the minor premiss nor the conclusion is true. The latter is

$$(1-k)(1-q) = 1-q-k+kq = 1-k\bar{q}-q = pq-q = -q\bar{p}.$$

Therefore $\bar{k}\bar{q}+q\bar{p} = 0$. This means that $\bar{k}$ is a negative quantity.

Now $1 > p$; $\therefore$ $q > pq$; and $1-\bar{q} > pq$. But $k\bar{q} < \bar{q}$; and $1-k\bar{q} > 1-\bar{q}$; $\therefore$ $1-k\bar{q} > pq$; $\therefore$ $1 > pq+k\bar{q}$. Hence absolute certainty could never be attained by such syntheses. Since $\bar{k}$ is negative $k > 1$; and $\bar{q}$ must be all the smaller, and q the greater. But k is never > 1.

Let us, however, assume that certainty has been attained. Then

$$1 = pq+k\bar{q}, \text{ and } 1-k\bar{q} = pq.$$

Now since k must lie between the limits 0 and 1:

(*a*) If $k = 0$; $1 = pq$; and both p and q must be $= 1$; that is the premisses must each be absolutely certain.

(*b*) If $k > 0 < 1$; that is any degree of probability. Since $1 - k\bar{q} > 1 - \bar{q}$, it follows that $1 - \bar{q} < pq$; or $q < pq$; $\therefore\ p > 1$; which is absurd.

(*c*) If $k = 1$; $1 - \bar{q} = pq$; or $q = pq$; $\therefore\ p = 1$.

Hence absolute certainty is possible only if either both premisses are absolutely certain; or else the major premiss is certain; although the minor only probable; and yet if not true, the conclusion be still true and certain independently of it; since $k = 1$ the intuitive or illative sense being certain although the minor premiss is anything but certain. In either case, however, the major premiss or universal must be certain.

Hence without a universal no certainty is possible.

Without a universal absolute knowledge is impossible.

The Hegelian dialectic therefore could never bring us to the Absolute unless we started with it. That is, the Absolute which it is the object of the Dialectic to attain must postulate the Absolute.

The Hegelian has as little chance by this dialectic of ever attaining the truth as the Pragmatist.

If we deny that we have any intuitive certain knowledge to start with, such as Descartes' 'Cogito ergo sum' or the knowledge of a Perfect Being, as in Chapter IX, knowledge must be admitted to be impossible so far as the Absolute is concerned; and only degrees of approximation are possible.

§ 5

The identity $x^n = x$ does give many and innumerable solutions which afford the dialectic full play. But there is one, and only one, that leads to truth. Yet this cannot be reached through the means of others. Only by starting with the Absolute $x = 1$ can a synthesis be built up, leading to further truth; though the actual materials in the minor premiss may be far from absolute certainties. They can by the aid of universals in the major premiss lead to absolute truth. But only by the aid of intuitive knowledge which gives it 'a leg up', so to speak. This is the Kantian theory of the basis of *a priori* synthesis of knowledge. The minor premiss supplies the knowledge derived from experience.

The constant k is the intuitive factor ascribed to faith. But the object of science is to dispense with it as far as possible; and, try as it may, it cannot do so if knowledge is to be attained.

In all science there is necessarily implied a certain element of faith or expectation which, disguise it as we may, asserts itself again and again as soon as an examination of the basis of our knowledge is attempted. 'There lives more faith in honest doubt,' said Tennyson; but I prefer to say, 'There lives more certainty in honest faith than in half the creeds and sciences'. For honest faith is the basis of all doubts, and there is no creed or doubt that is not the offspring of some lone purpose based on faith or expectation.

To dispense with this element in the process of reasoning by probabilities we must suppose that the conclusion in a hypothetical syllogism cannot be true if the minor premiss is false; i.e. $k = 0$, an assumption that has no logical ground whatsoever. For from truth nothing but truth can follow; but from error truth may follow. It is in fact Bacon's fatal fallacy in the *Novum Organum* and *De Augmentis Scientiarum*, that of ignoring the plurality of causes. This was emphasized by J. S. Mill in his *System of Logic*.

We have seen that when $p = 1$, and $k = 1$, then π, the probability of the conclusion, is 1. For $\pi = pq+k\bar{q} = q+\bar{q} = 1$. $\therefore \pi = 1$. The illustrations (§ 6) bring out the point of real practical importance in this problem; but they should not be regarded as expressing any particular views on the subject-matter, apart from the actual stress laid on the distinction between *a posteriori* and *a priori* probabilities.

We may first enter into greater detail with regard to the process of dialectic. Let us suppose that (1) the major premiss is first negated.

Then if r_1 represent the degree of truth attained by the process; $r = pq+k\bar{q}$, and $r_1 = c_1\bar{p}q+k\bar{q}$. And if R_1 be the degree of truth of a further negation of $c_1\bar{p}$, we have $R_1 = r+a$, the difference $= a$ and also $R_1 = (1-c_1\bar{p})q+k\bar{q}$; $\therefore a+pq+k\bar{q} = (1-c_1\bar{p})q+k\bar{q}$; and $\therefore$ we have $a = q\bar{p}\bar{c}_1$.

Consequently
$$R_1 = (p+\bar{p}\bar{c}_1)q+k\bar{q}. \qquad (4.08)$$

If the negation of this be

$$R_1' = c_2(1-p-\bar{p}\bar{c}_1)q+k\bar{q} = c_2c_1\bar{p}q+k\bar{q}.$$

Now
$$R_2 = \{p+\bar{p}(1-c_2c_1)\}q+k\bar{q};$$
$$R_2' = c_3\{1-p-\bar{p}(1-c_2c_1)\}\ q+k\bar{q},$$
$$= c_3c_2c_1\bar{p}q+k\bar{q};$$
$$R_3 = \{p+\bar{p}(1-c_3c_2c_1)\}\ q+k\bar{q};$$

$$R'_3 = c_1c_3c_2c_1\bar{p}q+k\bar{q};$$

. . . . . .

$$R_n = \{p+\bar{p}(1-c_nc_{n-1}c_{n-2}c_{n-3}\ldots c_2c_1)\}q+k\bar{q}. \qquad (4.09)$$

Hence if n is infinite $c_nc_{n-1}c_{n-2}\ldots c_3c_2c_1 = 0$.

$$R_n = q+k\bar{q};$$

which is the same as if $p = 1$.

That is, the result of the process of dialectic after an infinite number of negations is the same as if the major premiss had been a certainty; the probability of the minor being q, or constant throughout.

If, however, the major premiss had been kept constant and the minor negated through a number of similar reversals of negation and reaffirmations, we should have obtained by an analogous method of reasoning the following results:

(2) $r = pq+k\bar{q}$; $r_1 = c'_1p\bar{q}+\bar{k}q$; $R_1 = p(1-c'_1\bar{q})+k\bar{q}$;

and if

$$R_1 = r+b;\ b+pq = p(1-c'_1\bar{q});\ \text{or}\ b = p\bar{q}-pc'_1\bar{q} = p\bar{q}\bar{c}'_1,$$

therefore

$$R_1 = pq+k\bar{q}+p\bar{q}\bar{c}'_1 = p(q+\bar{q}\bar{c}'_1)+k\bar{q} = pq+(k+p\bar{c}'_1)\bar{q},$$

or $$R_1 = pq+k_1\bar{q},\ \text{where}\ k_1 = k+p\bar{c}'_1;$$

and it is easy to see that

$$R_n = pq+k_n\bar{q},\ \text{where}\ k_n = k+p\sum_1^n \bar{c}'_n. \qquad (4.10)$$

Hence if $\sum \bar{c}'_n = \bar{k}$, then $R_n = pq+(k+p\bar{k})\bar{q} = r+p\bar{k}\bar{q}$; since $r = pq+k\bar{q}$, and if $p = 1$ then $R_n = 1$.

If $k = 1-\dfrac{\bar{r}}{p\bar{q}}$, then $\bar{r} = p\bar{k}\bar{q}$ and $R_n = 1$. But it is easily seen that this implies $k > 1$ and is therefore impossible.*

Similarly in (1), if $\bar{p}(1-c_{n-1}c_{n-2}c_{n-3}\ldots c_2c_1)q = \bar{r}$; then $R_n = 1$, since $r = pq+k\bar{q}$ whatever be the values of p or q or k,

that is, if $$c_{n-1}c_{n-2}\ldots c_2c_1 = \frac{\bar{p}q-\bar{r}}{\bar{p}q} = 1-\frac{\bar{r}}{\bar{p}q}.$$

By substituting the values of $\bar{r}$ in terms of p, q, and k in this, it is found that it is always negative, so that R_n is never 1.

Hence it is not possible to arrive at a certainty by dialectic process on either the major or the minor.

* If $\bar{r} = p\bar{k}\bar{q}$ then $1-pq-k\bar{q} = p\bar{q}(1-k)$,

$k\bar{q}\bar{p} = 1-p(q+\bar{q}) = 1-p = \bar{p}$,

$\therefore$ $k\bar{q} = 1$ or $k>1$ since $\bar{q} < 1$.

§ 6

Some examples:*

Let X = 'Miracle recorded in the Bible.'
Y = 'Recorded statements in the Bible.'
Z = 'Translation of recorded statements in the Bible.'
W = 'Interpretation of translated statements in the Bible.'
S = 'Inspiration of Holy Scriptures, in regard to the particular passage.'

Put p = Probability that 'If Y is true X is true'. i.e. 'If the recorded statements are true the miracle took place as stated in the Bible'.

q = Probability that 'If Z is true Y is true'. Namely that 'If the translation is accurate the recorded statement is true'.

r = Prob. that 'If W is true Z is true'. i.e. 'If the interpretation of the alleged miracle is correct, the translation is accurate'.

s = Prob. that 'If the Holy Scriptures are inspired, the interpretation of the translated statement is correct'.

Then, as we have seen that t = the probability that if the Holy Scriptures are inspired the alleged miracles took place, is not as the sceptic might imagine merely $p.q.r.s$; but

$$t = s.(pqr + c_1\bar{q}r + c_2\bar{r}) + c_3\bar{s}.$$

To the sceptic the probability of the miracle is c_3, since $s = 0$.

But to the Christian the probability of the Inspiration is very much greater. Yet, if he thinks clearly, the certainty of the inspiration does not justify certainty, since the accuracy of the interpretation, translation, and traditional records have to be considered.

Again we may point out that in the case of the miracle of Christianity itself, its extraordinary rise in spite of the most determined opposition and persecution; and finally the conversion of the whole Roman Empire though regarded by Christian apologists as almost miraculous, if not actually so, has been accounted for by Gibbon by his now famous five natural causes, no one of which he admitted would have been sufficient to explain the facts; but when taken together could have produced the astounding result which the history of the Church attests as the most wonderful triumph of the prophecy that 'the gates of hell shall not prevail against her'.

Now Cardinal Newman in his *Grammar of Assent* replies to Gibbon that even if the probability of each one of the five causes be within the bounds of ordinary credibility, yet the chance of their combining is insignificantly small. Not so small however, I venture to think, as might at first be imagined. For even if that chance were an even one for each cause the probability of their conjunction would be $1/2^5 = 1/32$; a chance by no means as insignificant as Newman imagined and far from miraculous. That one such religion in thirty-two should triumph throughout the ages is not exactly a phenomenon of a miraculous nature, or if the chance of each cause be

* A miracle is supposed to be a supernatural intervention of the laws of Nature, and cannot, scientifically at least, be distinguished from an emergence, which is the coincidence of two complex or imaginary factors $\sqrt{-1}$ in a combined reality -1, as in the case of time. One could hardly regard time as a miracle, unless all Nature is such, and that I am quite willing to grant.

$\frac{1}{10}$ the chance of their combined action $= \frac{1}{10^5}$. An event of one in a hundred thousand could hardly be deemed to be miraculous though wonderful. It is the chance that one person out of a thousand should live to a hundred.

As a matter of fact the probability is even greater than this, for if we take into account, as we should do, the probability of the triumph of Christianity through or in spite of the absence of any one of the five causes given by Gibbon; small though these probabilities might be, their sum in the manner just indicated for the hypothetical sorites would add, and under certain circumstances, add considerably, to the probability of the result. These circumstances would be similar to those indicated in (4′) only more fully developed to five constants c_1, c_2, c_3, etc.

It must be borne in mind, however, that although the actual probability of the spiritual origin of Christianity is undoubtedly diminished in view of these considerations, yet there is nothing to prevent an extra premiss from being added involving a term c_6, such that the probability of the eventful rise of Christianity independently of all natural causes, owing, for instance, to the spiritual nature of its teaching and the direct appeal it makes to the religious sense of mankind, raises the probability to a considerable credibility.

It is from considerations such as these, perhaps, that men of the mathematical stamp and high calibre of Dr. Salmon and Dr. J. H. Bernard were led to take up theology, all through the element of probability of a personal nature based upon the religious sense. And the extent to which this element of probability depends upon the individual is fully realized in the respect with which differences of opinion are now tolerated, notwithstanding the earnestness of conviction of each adherent to his own particular faith or belief or unbelief. The probability in its ultimate aspect is indefinable, but profound, and remains a factor that the serious thinker cannot neglect without the taint of unscrupulous prejudices.

The five causes assigned by Gibbon are (1) the zeal of Christians inherited from the Jews, (2) their doctrine of a future state, (3) their claim to miraculous powers, (4) their virtues, and (5) their ecclesiastical organization. To these, however, may be added (6) the influence of authority supported or asserted by a Divine Revelation playing upon the most natural and primitive instincts of fear, and (7) the apparently consistent and coherent teaching which afforded at once a philosophy to the religious instincts and philosophical cravings of ordinary and often extraordinary men, and still more perhaps of women too; for the latter may be said to be the backbone of religion. Religion is not necessarily the philosophy of the ignorant as Averroës has described it, but it may roughly be regarded as the philosophy of the emotional. Yet not strictly speaking so, for philosophy is rational, and religion is real whether rational or not. It is no more philosophy than music and appeals to the primitive instincts like art as a living power, and short path to reality. The appeal of Christianity then has been that it

seemed to be the *real thing* which satisfied the aspiration towards the unknown, and afforded a perfect relationship of man with the mystery of a world beyond.*

§ 7

Let p = Prob. that 'If an interpretation of a certain passage in the Scriptures is accurate and correct, the idea embodied therein is an inspired truth'.

q = Prob. 'A's private judgement is accurate and correct'.

π = Prob. 'A's private judgement accords with inspired truth'.

k = Prob. that if A's private judgement be inaccurate, it may still accord with inspired truth; for although his judgement may be wrong as regards the particular matter, he may give his assent contrary to his own judgement in accordance with some higher authority that might be right.

Then as before $\pi = pq+k\bar{q}$;

If the Scriptures are inspired, we may take $p = 1$, so that

$$\pi_1 = q+k\bar{q};$$

Hence when $k = 1$ is π a maximum and equal to 1 since $q+\bar{q} = 1$. Of course k may never be 1, as Protestants hold. The inspiration therefore cannot make $\pi = 1$ unless $q = 1$. This would be assuming the individual himself to be infallible. Only if $k = 1$ does π attain its full value and significance, because of $1 = q+k\bar{q}\ \bar{q} = k\bar{q}$ and therefore $k = 1$.

It might be supposed that the constant k leads on to all sorts of credulity, superstition, and so forth. But this is not necessarily so. For if there is certainty about anything, $k = 0$, or $= 1$. Therefore either $pq = 1$, i.e. $p = 1$, and $q = 1$; or $pq+\bar{q} = 1$. The latter means $pq = q$. $\therefore p = 1$.

Hence in either case, if the inspired truth is a certainty, the Holy Scriptures must be interpreted accurately. But in one case the individual must himself be infallible in his private judgement, at any rate as regards his interpretation of the Scriptures: in the other case though fallible in his own private judgement, his assent through a higher authority which is itself infallible gives him the assurance of certainty. Without such an authority, therefore, he can have no basis of certainty.

This was the line of argument taken up by Duns Scotus, the

* The proper attitude towards Christianity is that taken up by Fichte, Schelling, and Schleiermacher, and to a large extent by Hegel. The true basis and value consist in its teaching and philosophy, rather than in its historical foundation and appeal to the miraculous.

Doctor Subtilis. To him, the creation, the immortality of the human soul, and even the existence of God Himself admitted of no definite proof, except on the assumption of the authority of the Universal Church.

Newman adopts a similar line of argument in his *Grammar of Assent*, and the Earl of Balfour, in *Philosophic Doubt* and the *Foundations of Belief*, pursues a course which, I think, leaves no doubt that for him $k = 1$; though not falling back upon the Church, he nevertheless postulates an assent through belief as the basis of sane reasoning. It must not be forgotten, however, that Balfour and Newman virtually implied Kant's transcendentalism. Though Balfour after Kant has recourse to the transcendental ideas of the pure reason, Newman after Scotus takes refuge in the bosom of the Catholic Church. For the latter there is no alternative between atheism and Catholicism. Kant's belief in God was based upon the Moral Law, and nothing that conflicted with his idea of it was acceptable as true. Both base the belief in God on the moral sense as a major premiss; just as one might accept the authority of the Scriptures, as a revelation. But they differ once more in the minor premiss; for Newman, though basing his belief upon the moral sense, denied that he could find any Moral Law, as regards both broad principles and details, outside the Catholic Church. Though conscience is the basis of morals, one's ideas of right and wrong might be poles apart from those of others. Nietzsche might be regarded as acting conscientiously and probably did do so. The sense of beauty could be identified, if one thought right, with the sense of good. And even Oscar Wilde might be regarded as a saint.

The common sense of most educated men in northern Europe, however, has rebelled against the supposed rigour and discipline of the Church. Kant having based his belief in God on the Moral Law found within him, could accept nothing that seemed to contradict it. If therefore there is no infallible authority, we must work upon the average common sense of mankind, with no very definite standard; a somewhat variable and shifty ground, no certainty being possible. All that can be said is that for the Catholic $k = 1$; and for the Protestant $k = 0$. One feels sure of his ground, the other not so. He who feels sure of this world and certain of the next need have no fears of a hereafter, save of the thorns that in his bosom lie to prick and sting him as in some tight array. But the other in whom the sense of right and wrong is no less developed, though a law unto himself can feel perhaps at times a no less discomforting apprehension The transcendental

pure reason drives one into the Church and the other out of it, according to the natural disposition of his mind and character.*

It would be idle to pursue this particular illustration further, so far as the theory of probabilities in respect to the logic of the hypothetical syllogism may be concerned; or the Hegelian dialectic of approximate truths. It is hardly fashionable. The antithesis can be overcome only by the formation of a higher synthesis.

In the absence of faith or intuition, the conclusion can never reach a degree of certainty unless the minor premiss is as certain as the major universal.

According to Aristotle, as we have seen, the universal is true only in the particular. So that $p = 1$ only if $q = 1$; or both p and $k = 1$. If $p = k = 1$, then the intuitive perception of the universal as the truth renders the conclusion certain, even if the particular which contains it is far from certain. This is a most remarkable result, and one that may be at first sight not quite obvious. But if the universal must, or can, be true independently of the particular, it takes the basis out of the Aristotelian doctrine. If knowledge is possible and at the same time dependent upon intuition, as Kant held, no amount of experience could ever by mere synthesis or otherwise acquire a full measure of certainty; and truth therefore is unattainable.

Writers on this subject employ a diffuse and vague terminology coupled with endless circumlocution. When Goethe exclaimed 'I have no word for it; feeling is all!' he may have had in mind either p, or q, or k; and most probably some combination of all three. We have seen that in Bradley's system the element of feeling enters as an essential factor, and renders the dead, or at any rate lifeless, Aristotelian proposition or judgement a living reality, but permanent truth; at once full of the endless meaning and variety of ideas that the identity in the universal necessarily implies, and as a complete logical process by necessity entails, in the perfect realization of the Totality in the Absolute. For how else is knowledge other than merely sensuous perception possible? or the empty formula to be endowed with the living sense of reality in the self-perception of Truth?

Not only thought and feeling enter into this conception of the truth but will in action, as conation. The merely probable can

* A classical scholar who did not quite follow the above argument wittily suggested that without doubt the situation might be summed up in a truly Chestertonian manner by asserting that the Protestant believes his judgement to be A.1; whilst the Catholic believes the Pope and the Church to be O.K. The one is sure of himself; the other is certain he is on safe ground.

never lead to decisive and definite results. Action is the result of purposeful feeling.*

'The notion in Hegelian philosophy is the principle of freedom; the power of substance self-realized. It is the principle of all life, and thus possesses at once the character of thorough concreteness. In short, it is what contains all the categories of thought merged in it. It is a form, but an infinite and creative form, which includes, but at the same time releases from itself, the fulness of all content.' The notion is a true concrete; for it involves being and essence, and the total weight of these two spheres with them merged in the unity of thought. For the Absolute is the notion. The full development of the notion is effected, as we have seen, not through any inherent idea like a Chinese puzzle trick of boxes within boxes; but by a process of 'emergence', an expansion of terms in the manner we have indicated through the insertion of the intuitions of space and time; and the emergence of real terms from the combination of imaginary ones as in the theory of Relativity; already shown to lead to action from the combination of 'cognition' and 'affection' or feeling in time.

§ 8

In the calculation of probabilities and in the principle of induction, which enables us to anticipate from a number of occurrences of an event its further recurrence, are to be found a basis of experience. Thus if an event occur m times in succession, what is the probability that it will occur $(m+1)$ times? The problem is equivalent to Laplace's well-known determination of the probability of drawing a red ball from a bag containing an unknown number of balls. m balls having been drawn in succession, the probability that the $(m+1)$-th ball will be red, assuming that the total number of balls is infinite, or else that each ball is replaced after being drawn, and that there are only two colours, is $= \dfrac{(m+1)}{(m+2)}$. (4.11)

If, however, there are n colours, the probability is $= \dfrac{2.(m+1)}{n(m+2)}$.

Let p be the probability of drawing a red ball if there are but two colours; then if the chance of drawing a red ball is the same as that of the other, say blue, $p = 1/2$. And if there be n colours all

* Are you in earnest? Seize this very minute,
What you can do, or dream you can, begin it.
Boldness has genius, power and magic in it,
Only engage and then the mind grows heated—
Begin, and then the work will be completed. (*Faust.*)
The idea here is to act, in the belief that action leads to thought.

equally probable, $p_n = 1/n$. Therefore $p_n/p = 2/n$, or $p_n = \frac{2}{n}p$. The probability that a red ball should be drawn m times in succession is $p_n^m = \left(\frac{2}{n}p\right)^m$. If now c is the probability that p should lie between $p+dp$ and p, and supposing that all probabilities are possible between 1 and 0, the probability of drawing m red balls in succession is

$$= c^m \int_0^1 \left(\frac{2p}{n}\right)^m .dp = c^m \cdot \frac{2^m}{n^m} \frac{1}{(m+1)} .$$

c is the probability that the probability of drawing a red ball lies between $p+dp$ and p and is very small.

Also likewise the probability for $(m+1)$

$$= c^{m+1} \cdot \frac{2^{m+1}}{n^{m+1}} \cdot \frac{1}{(m+2)} .$$

Consequently the probability that if m occur, $m+1$ will likewise do so is, of course, the probability of the latter divided by that of the former.

Hence the probability that if m red balls have been drawn, the $(m+1)$th will also be red, is $= \frac{2c}{n} \cdot \frac{(m+1)}{(m+2)}$.* (4.12)

It will be observed that in an ordinary chance event of probability p, the probability of the event happening m times in succession being p^m, and $m+1$ times accordingly p^{m+1}, it follows that the probability that having happened m times it will happen $m+1$ times is

$$= \frac{p^{m+1}}{p^m} = p. \quad (4.13)$$

There is thus a very great difference between the two problems.

The reason that the probability increases with the number of times the event has been repeated in the previous case is that the probability of the combined event may range anywhere between 1 and 0, whereas in the latter it is something definite.

The expectation of the recurrence of light and darkness, as in the case of the rising of the sun, comes under the problem of drawing m balls when there are only two alternatives, say white or black. The probability is then $\frac{m+1}{m+2}$, as there are but two alternatives.

* Dr. Broad has given an analysis of this problem based upon Mr. J. M. Keynes's theory, see *Proc. Arist. Soc.*, 1928. I give Boole's treatment after Laplace, though the constant c is omitted by them.

The probability of such an event increases from day to day, thus from 1/2, 2/3, 3/4, 4/5, 5/6, etc. The uniformity of Nature thus readily becomes a practical certainty.

In a similar manner the probability that an event that has happened m times in succession should not happen the $(m+1)$-th time is

$$1-\frac{(m+1)}{(m+2)}=\frac{1}{(m+2)}.$$

The probabilities are 1/2, 1/3, 1/4, 1/5, 1/6, etc.

The expectation therefore increases very rapidly with the number of successive occurrences. It might appear at first sight from this alone that Hume was right when he asserted that no amount of human evidence could prove a miracle; for it is always far more probable that the witnesses are deceived than that the order of Nature should be violated. But Hume was nevertheless wrong, as we have seen.

The probability that the sun would rise and set in the usual way is $\frac{m+1}{m+2}$, and the probability that it should not do so is $\frac{1}{m+2}$. This is not the probability to one unacquainted with the facts and principles of astronomy. Nevertheless, if we take the age of the Earth to be a thousand million years, and grant that the sun has risen regularly throughout that period, the probability that it should not rise; or what is virtually the same thing for our purpose that there was a violation of the order of Nature, would be $= 1/\ 365\times10^9 = 1/3.65\times10^{11}$; and therefore it would according to calculation be a credible phenomenon if attested by twelve reliable witnesses whose credibility was 10/1. (See p. 224.) It must not, however, be supposed that the miracle recorded in the Book of Joshua would be of this order; for the probability of the various other premisses should be considered, and as we have seen the evidence is nth hand. As scientific evidence it is of course negligible; but if the credibility of the Bible is not that of an ordinary witness, nor a group of witnesses, but of an inspired work, the value of its testimony might rise approximately to unity to a person who believed in it as such.

The point to be emphasized is that Hume was entirely wrong in asserting that no amount of human testimony could prove a miracle. He showed no acquaintance with the real difficulties in the question, giving vent to an unconquerable prejudice, for the probability of the truth of an assertion increases in geometrical progression, as the number of independent witnesses increases in arithmetical progression.

If the testimony of a highly-trained scientific observer be, let

us say, $10^4/1$, the testimony of only three such witnesses would satisfy the limit of ordinary credibility. An event so asserted should be regarded as a discovery of an unaccountable phenomenon; for instance, the discovery of the X-rays, or that of radium when first announced. If the record of the Book of Joshua is accurate, there must have been a suspension of all the laws of motion; and if the Earth had suddenly ceased to rotate the effects of inertia would have been terrific and disastrous throughout the whole globe; its consequences would have been manifest as a geological catastrophe shattering the Earth. Else all Nature must have been temporarily changed and as such a miracle would obviously be the only explanation. The probability of this would be undoubtedly very much less than that of the sun not rising. Yet even if the chance that the whole order of Nature had been suspended be the billionth of that of the sun not rising; and if the number of independent credible witnesses be doubled, the credibility of the event becomes quite ordinary and well within the limits of reasonable belief as before. Thus the testimony of twelve scientific men or trained observers whose credibility, if taken as low as $100/1$, would still be sufficient to render credible an event the credibility of which is only $1/10^{22}$. But this is assuming what is not at all likely, that their credibility is unaffected by the opinions of others. Even a crowd of trained experts can be swayed by its own dominant opinion or that of one of its members, and the credibility of each of its members may depend upon the nature of the event in relation to their prejudices. However unlikely therefore that the accepted order of Nature could be proved, on mere human testimony, to be reversed; the possibility of miracles as extremely unlikely phenomena cannot be ruled out of court in any strictly scientific inquiry. It is only after all possible natural explanations have failed to give an adequate explanation of the facts, if these shall have been satisfactorily established, that a supernatural theory can be employed. For the principle of the continuity of Nature is the best established generalization in modern science. Unless as in the miracle of Joshua the whole order of Nature be suspended, the assumption that merely an abnormal phenomenon had taken place would be the explanation acceptable to a scientific mind, in preference to a supernatural intervention. Many religious enthusiasts are much too prone to accept all sorts of miraculous explanations on much too slender evidence. Moreover, statements alleging facts of such immense improbability naturally tend to discredit the witnesses. Unless their own predilections can be shown not to have affected their credibility, this is obviously

diminished. Further, the influence of suggestion such as that referred to in p. 53 has to be taken into account, so that the credibility of otherwise credible persons would be rendered almost nil. Nevertheless even this latter effect can at times be tested.*

* Miracles though possible and rendered probable on human testimony should be regarded scientifically as *ultra vires*, since the possibility of hitherto undiscovered laws and exceptional residual phenomena always remains until they have been eliminated.

CHAPTER XIV

THE PROCESS OF INDUCTION AND ANALOGY

§ 1

THE process of induction is the equivalent of a thorough analogy, where the connotation and the denotation have been reversed or substituted for each other. Thus if two or more things have a large number of qualities in common, then that they will resemble each other in all other qualities is an argument from analogy; but it is really an induction as regards the qualities in question, that is the connotation; for if we were to substitute the denotation for the connotation in the argument, there would be no question as to its being an induction. Thus, if two or more qualities are invariably found in a large number of objects, they will coexist in all bodies. This is an induction differing in no way from the former except that the generalization refers to objects instead of to qualities. It must be remembered, however, that, as a rule, an analogy is a very incomplete induction as regards qualities; and if denotation were substituted for connotation the induction would be regarded as so weak that it might almost rank with a superstition. Thus analogy should be considered philosophically as of little value unless it admits of being regarded as an induction of connotation. In such a case it would rank high in scientific speculation, in precisely the same manner as the usual inductive arguments employed in scientific method.

Unfortunately, analogy as a method of research has perhaps been unduly discredited on account of the use made of it by such writers as Bishop Butler in the *Analogy of Religion* in the past. This book aroused the reverence and enthusiasm of many scholars, who took it seriously and regarded it as of real value. Yet 'Analogy', as Darwin said, 'is a deceitful guide'. Looked at, not in the old-fashioned illicit generalization of qualities, but as a proper induction in which connotation takes the place of denotation as the subject of induction, it should be taken seriously as a scientific mode of argument. The 'illative sense' is possibly but the power of generalizing with regard to connotation. It is not a matter of 'intuition', but rather one of induction of connotation, which, as has just been shown, is in fact an analogy of the perfect type. The supposed intuition is merely the result of rapid generalization when not due to bias or unconscious prejudice.

Quickness and clearness of perception are often, and indeed usually, nothing more than the result of rapid association and induction. The method here employed is largely of this nature, whilst the actual details in regard to the phenomena of Nature come under physics and kindred subjects. But the general principles are dealt with as a branch of the more general science of logic.*

§ 2

This mysterious contradiction of the dialectic is nothing else than a recognition that a high degree of similarity in connotation is still far removed from an identity, and a high degree of probability in denotation yet removed from certainty. The dialectic logic is thus the equivalent of the theory of probabilities, and by no means to be overlooked by the scientific man. It is, in fact, by the recognition of the striking resemblance between them and their combined use that the results put forward have been arrived at and evolved. Probability and variation go hand in hand, and reason is the transcendence of chance, or that which makes chance possible.

In dealing with the dialectic, as already shown, the negation of a negation does not necessarily bring us back to the original idea; for we are, it is admitted, dealing with slightly different things as though they were the same. The apparent identity in apparently the same connotation is apt to lead one astray, for identity in $(n-1)$ qualities, even where n is a very large number, may still mean a difference in the nth that enables us to distinguish between them; and hence X becomes not X, although strictly speaking neither is the other. If E represents the essence of a thing, or idea, and $x_1, x_2, \ldots, x_n$ the accidents, then if two things have every quality $x_1, x_2, \ldots, x_{n-1}$ in common, but differ as regards x_n, the two

* The problem before us is not how the synthetic production of life is to be effected at the present day, for I have again and again endeavoured to show that anything produced to-day as a parvenu must differ so immensely from its contemporary with myriads of generations of ancestors, that a new definition of life would be necessary in order to identify them as the same. This I endeavoured to do, somewhat prematurely perhaps, but none the less suggestively, in the *Origin of Life* as far as emphasizing this point and removing the misconception. But the same old talk goes on by the same old wiseacres. 'You cannot produce life at all,' they say, or, 'You will perhaps produce it some day, but you have not done so yet!' I can only reply once more that 'If I did produce life to-day, none of them would admit it was anything like the life around us, any more than a sperm or an ovum is like a man, and if it was similar enough to be recognized as such, I for one would not admit it had been produced artificially, and would rather believe there had been some mistake'. By the time this idea shall have pierced the sconce of one or two of my most supercilious critics the problem will no doubt have been solved.

cannot be the same, however much they may resemble each other in all other respects. Their connotation may be expressed by

$$E.x_1x_2x_3 \ldots x_{n-2}x_{n-1}x_n,$$
$$E.x_1x_2x_3 \ldots x_{n-2}x_{n-1}\bar{x}_n. \qquad (4.14)$$

A further negative of the latter is

$$E.x_1x_2x_3 \ldots x_{n-2}\bar{x}_{n-1}\bar{x}_n.$$

This process may be repeated until we get back to E, assuming of course that there is such a thing as E. There must ultimately be a bedrock, or something which is identical with itself, else all reasoning becomes illusory. It is then subject and predicate identically in one, the *notion* as Hegels calls it, as distinct from an *idea* which is the notion and the object in one. The analogy between the theory of probabilities and the illative sense holds only in regard to groups of qualities, and not to the relations between individual qualities themselves. This distinction should be emphasized lest any misapprehension should arise upon this point.

If the probability of an extremely unlikely event happening, such as the existing order of creation from chaos, becomes greater and greater as its order settles down, the improbability of such events recurring becomes less and less, until the probability of their happening is so great that the expectation becomes a 'moral certainty', as in the case of the sun rising or the drawing of red and white balls from a bag. Here, however, the 'certainty' is due to the denotation, whereas in the case of the former to the connotation, call it the 'illative sense', the Hegelian 'notion', the certainty is moral and self-evident, as the result of the elimination of the irrelevant.

The 'Notion' corresponds to Kant's 'Idea' of the transcendental reason which does not contradict itself because above phenomena. The belief in the 'Uniformity of Nature' is based upon the probability of the denotation, but the belief in reason is based upon connotation, for certainty is nothing more than the elimination of doubt through the synthesis of dialectic, as distinct from 'expectation' which is the intuition of the understanding.

The transcendental ideas of Kant do not follow from the denotation but from the connotation. And yet only after the intuitions of the understanding, namely the categories, have been transcended. In the general expression

$$(x_1+\bar{x}_1)(x_2+\bar{x}_2) \ldots (x_n+\bar{x}_n) = 1,$$

the x_1, x_2, ... may be used for the categories and the sensuous intuitions of space and time, but the transcendental ideas which

are independent of experience, still come after them in the expansion, as the full expression of the universe of discourse. Thus the transcendental ideas bear the same relation to the categories and the sensuous intuitions as these do to the objects of the understanding and the senses, presupposing them and rendering them possible.*

Hegel very rightly points out, as already emphasized, that the twelve categories of Kant might well have been extended to an indefinite number. But the classification adopted by Kant has many advantages quite apart from those Hegel had in mind. The classification is far from exhaustive, but nevertheless suggestive, whether we actually have recourse to the Kantian, Aristotelian, Hamiltonian, or other set of categories we might happen to employ.

* The reader who wishes to become fully acquainted with the principles of the critical philosophy should read Kant in the original, or a translation, of which there are Mahaffy and Bernard's excellent editions of the *Kritik* and the *Prolegomena*, and Meiklejohn's admirable translation of the former in Bohn's Library.

CHAPTER XV

THE THEORY OF EMERGENCE

§ 1

THE idea of *emergent* evolution now obviously comes within the range of the logical concept, unexpected as this might at first sight have been; for the identity which is distinguishable in differences is the outcome of the higher laws of thought, as implied in the algebraic expression of identity, in a manner which only the mathematical form adequately conveys.

The emergence is a variation which, though continuous in one scale, can present the appearance of discontinuity or mutation in a higher scale; as I have already endeavoured to make clear.

Thus if

$$1 = \bar{x}.\bar{y}.\bar{z}.\bar{w}+x.\bar{y}.\bar{z}.\bar{w}+x.y.\bar{z}.\bar{w}+x.y.z.\bar{w}+x.y.z.w,$$

this is the direct outcome of the relations

$$w.\bar{z} = 0,\ z.\bar{y} = 0,\ y.\bar{x} = 0, \qquad (4.15)$$

which combined may be written

$$y.\bar{x}+z.\bar{y}+w.\bar{z} = V.*$$

Now putting x, y, z, and w, equal to 1, and 0, respectively, we obtain

$$\begin{aligned} V = \quad & 0.x.y.z.w+0.x.y.z.\bar{w}+1.x.y.\bar{z}.w+0.x.y.\bar{z}.\bar{w}+ \\ & +1.x.\bar{y}.z.w+1.x.\bar{y}.z.\bar{w}+1.x.\bar{y}.\bar{z}.w+0.x.\bar{y}.\bar{z}.\bar{w}+ \\ & +1.\bar{x}.y.z.w+1.\bar{x}.y.z.\bar{w}+1.\bar{x}.y.\bar{z}.w+1.\bar{x}.y.\bar{z}.\bar{w}+ \\ & +1.\bar{x}.\bar{y}.z.w+1.\bar{x}.\bar{y}.z.\bar{w}+1.\bar{x}.\bar{y}.\bar{z}.w+0.\bar{x}.\bar{y}.\bar{z}.\bar{w}. \end{aligned}$$

Hence

$$x.y.z.w+x.y.z.\bar{w}+x.y.\bar{z}.\bar{w}+x.\bar{y}.\bar{z}.\bar{w}+\bar{x}.\bar{y}.\bar{z}.\bar{w} = 1.$$

Otherwise we have in general, as previously:

$$(x+\bar{x}).(y+\bar{y}).(z+\bar{z}).(w+\bar{w}) = 1;$$

and since

$$y.\bar{x} = 0; \quad (xy+x\bar{y}+\bar{x}\bar{y}).(z+\bar{z}).(w+\bar{w}) = 1;$$

and $z\bar{y} = 0$; $\quad (xyz+\bar{z}(xy+x\bar{y}+\bar{x}\bar{y}).(w+\bar{w}) = 1;$

also $\quad w\bar{z} = 0$; $\quad (xyzw+\bar{w}(xyz+\bar{z}(xy+\bar{y}(x+\bar{x})))) = 1.$

That is, as before,

$$xyzw+xyz\bar{w}+xy\bar{z}\bar{w}+x\bar{y}\bar{z}\bar{w}+\bar{x}\bar{y}\bar{z}\bar{w} = 1.$$

* Or if $w\bar{z} = 0$; $wz\bar{y} = 0$; $wzy\bar{x} = 0$, see p. 213.

Thus in general if $x_1, x_2, x_3, \ldots, x_n$ be the n terms under discussion

$$x_1x_2 \ldots x_n + x_1x_2 \ldots \bar{x}_n + x_1x_2x_3 \ldots \bar{x}_{n-1}\bar{x}_n + \ldots + x_1\bar{x}_2 \ldots \bar{x}_n + \bar{x}_1\bar{x}_2 \ldots \bar{x}_n = 1,$$

the premisses being (where r has all values from $1 \ldots n-1$) $x_{r+1}\bar{x}_r = 0$; so that there are $(n-1)$ relations between the n terms. If n is infinite the series is continuous. It is in fact the infinite possible variety of terms that makes them merge one into the other, and renders the conception of continuity possible.

When n is infinite, there would appear to be continuity between each pair of consecutive terms, and so ultimately of the whole, so that $x_1x_2x_3 \ldots x_n$ and $\bar{x}_1\bar{x}_2\bar{x}_3 \ldots \bar{x}_n$ may be regarded as continuous.

Now if $x_1 = x_2 = x_3 = \ldots = x_n$; then x_n is continuous with $(\bar{x})^n$; and x with $\bar{x}$, though $x + \bar{x} = 1$, or $x = 1 - \bar{x}$. If $(\bar{x})^n . x = 0$, then $x . x^n = x$, or $x . (x^n - 1) = 0$. The roots of this equation, as we have seen, are either $x = 1$, $x = 0$, or else negative or imaginary. Only for $x = 1$ is $x . (1 - x) = 0$. For $x = -1$, $x(1 - x)$ is not $= 0$, but only $x(1 - x^2) = 0$, and so on with the other roots of $x^n - 1 = 0$.

If x, y, z, w represent the connotation of a term or idea, i.e. its attributes, and X, Y, Z, W, its denotation or objects included by it; then we may write

$$xyzw + xyz\bar{w} + xy\bar{z}\bar{w} + x\bar{y}\bar{z}\bar{w} + \bar{x}\bar{y}\bar{z}\bar{w} = 1.$$

Also

$$\bar{X}\bar{Y}Z\bar{W} + \bar{X}\bar{Y}ZW + \bar{X}\bar{Y}\bar{Z}W + \bar{X}YZW + XYZW = 1.$$

If, further, as might appear obvious, though erroneously so, we regard the individual terms corresponding to each other as equivalent, then

$$xyzw = \bar{X}\bar{Y}Z\bar{W};\ xyz\bar{w} = \bar{X}\bar{Y}ZW;\ xy\bar{z}\bar{w} = \bar{X}\bar{Y}\bar{Z}W;$$
$$x\bar{y}\bar{z}\bar{w} = \bar{X}YZW;\ \bar{x}\bar{y}\bar{z}\bar{w} = XYZW.$$

This of course, leads to an absurdity, for the presence of the full connotation would mean the absence of all denotation. If, however,

$$xyzw = 0;\ \bar{X}\bar{Y}Z\bar{W} = 0;\ \bar{x}\bar{y}\bar{z}\bar{w} = 0;\ \text{and}\ XYZW = 0,$$

the remaining terms can be made to correspond, so that as the connotation becomes narrowed, the denotation is enlarged, and vice versa. If now we have $x_1, x_2, x_3, \ldots, x_n$, for the connotation, and $X_1, X_2, X_3, \ldots, X_n$, for the denotation, and n is a very large number, the same remarks will apply. And if n is infinite, then, that which has infinite qualities has no denotation, i.e. does not exist, whereas that which has no qualities at all, must have an infinite denotation. That is, the infinite has no qualities at all and is therefore nothing. Hence infinite connotation means no denota-

tion and infinite denotation no connotation whatever. This can easily be seen to be a fallacy.

The assumptions, though apparently symmetrical and homogeneous, are in fact entirely fallacious. The difficulty is overcome by supposing that an X_{n+1} stands for

$$\begin{array}{lll} x_1x_2 \ldots x_n, & \text{that is} & \bar{X}_1\bar{X}_2 \ldots \bar{X}_nX_{n+1} \\ \ldots\ldots\ldots & & \ldots\ldots\ldots\ldots \\ \bar{x}_1\bar{x}_2 \ldots \bar{x}_n & & X_1X_2 \ldots X_nX_{n+1}. \end{array} \qquad (4.16)$$

Hence if that which has infinite connotation exist as something, then that which has no connotation at all is something, namely the infinite totality of things, but it includes the term X_{n+1}.

It will be borne in mind that the terms are supposed to be one within the other, and related as *genera* and *species*, as the equations (4.15) indicate, otherwise the remarks would not follow. $x_n\bar{x}_{n-1} = 0$ but $X_1X_2 = 0$, $X_{n-1}X_n = 0$. There is something in common between the infinite totality of denotation, with no connotation whatsoever, and that unity without any denotation but itself, which embraces infinite connotation; the two, however, connotation and denotation, do not vary inversely as each other, as is sometimes popularly supposed.

Each term is a predicate or a subject to those before it, according to the order in which they are taken, and the relationship of subject and predicate is reversed when we pass from the consideration of connotation to that of denotation. It must be made unequivocably clear that $\bar{X}$ does not stand for x in any other than a 'parallel' sense; and that it is not its contradictory, or contrary. In other words, it cannot be said that $X+x = 1$, or that $X = 1-x$.

We can pick out any one term of either of these two series and we shall find that it contains all the classes within it on one side and is contained in all the classes on the other. The order of the two series is reversed according as we are dealing with connotation or denotation.

It will be observed that there is no fixed scale, standard or unit according to which this series, or rather either of these series could be divided up. We must indeed group them in the Gaussian fashion. As Bernard Bosanquet took so much pains to point out, the subject and predicate do not build up, but are parts of a judgement; or as Mr. Stout has well said, 'We are always developing a "subject" which is in our minds, and this "subject" is some region or province of the world of reality'. 'Every Judgement', says Bosanquet (see *Essentials of Logic*, p. 109), 'is a connexion of parts in a whole, and to be a whole is the characteristic of reality.'

'The natural course of thought is to define further what is already in great part defined, and our real world is that which we have so far defined.' 'Judgement is always the analysis and synthesis of elements in some one thing or ideal content. . . . It is not two things or two terms—but one thing or one term drawn out into elements by discriminating selection.'

Now the aggregation of qualities, or concentration of connotation, is what constitutes emergence and variation. The classes of ideas whether in connotation or denotation cannot be divided up into equal units, for there are no such units, and no means of defining equality. Quantity may not enter into the classification at all. But according to the Gaussian system of coordinates it is possible to classify events, or any groups of ideas, in such a manner as to dispense with the idea of quantity, so long as the 'order' of succession whether in time, or space, or logical sequence, is observed.

So long as the constituents of the connotative and the denotative series exist 'parallel', as it were, in time, things remain unaltered. But if through some fluke the steady state is slightly changed, so that the connotative series, let us say, gains in parts on the denotative; then a concentration of connotation would ensue, with a corresponding narrowing of the denotation; the point is that the relative positions of connotation and denotation may be altered at the same time or in the same space. Just as a vital force in matter could arise, as we have shown, from an unequal change in the dimensions of one particular kind of matter relatively to others around it, so could the emergence of new and unexpected qualities or attributes become manifest in the concentration of connotation in the logical series, relatively to that of the denotation through the mere *difference in the rate of progression of the event in time and space.*

When the Gaussian units are considered in regard to denotation, there would be a concentration in the individual, and variation or emergence would ensue. If, however, the Gaussian units are taken in regard to connotation, the change in denotation corresponds to the emergent quality in reproduction. The classes x and X no longer correspond, and an X may include a larger number of x's, or the x's a larger number of X's.

This seems to be the most promising method of approaching the problem of reproduction, for nothing else in physical conceptions enables us to perceive the possibility of apparently the same substance being reproduced *ad infinitum*! It is in truth all due to

the absence of pre-established harmony, and the grouping of monads round approximately equal, though never actually equal times.

Vice versa, if connotation and denotation can be altered in the logical series of judgements, in a given time, owing to the unequal progression of ideas in time, as the theory of relativity requires, then a change in these would be accompanied by a change in time, and accordingly with a change relatively to other bodies in space, on the lines indicated in Chapter XIX. Thought and will, or cognition and conation, once more become closely and intimately associated and correlated in such a manner that one may be regarded as but the expression of the other, or in fact as the other 'seen from another side'.

§ 2

The continuity in the logical series of judgements, one within the other, becomes as we have seen the outcome of the law of identity.

In the simple case of $(x+\bar{x}).(y+\bar{y}) = 1$, or $(xy+x\bar{y}+\bar{x}y+\bar{x}\bar{y}=1)$, since the actual arithmetical coefficients do not enter into the logical equations, so that the classes may be taken twice over, we may write $\bar{y}.(\bar{x}+x)+x.(\bar{y}+y)+y.(x+\bar{x})+\bar{x}.(y+\bar{y}) = 1$,

otherwise

$$\bar{x}.\widehat{\bar{y}+\bar{y}}.\underbrace{x+x}.\widehat{y+y}.\underbrace{x+x}.\widehat{\bar{y}+\bar{y}}.\underbrace{\bar{x}+\bar{x}}.\widehat{y+y}.\bar{x} = 1, \quad (4.17)$$

with $\bar{x} = \bar{x}$ uniting the two ends.

This of course merely gives symmetry to the equation without surmounting the real difficulty as to continuity, for the discontinuity between x and $\bar{x}$ and y and $\bar{y}$ still remains.

If we suppose, however, that $x.\bar{x}$ is not $= 0$, unless it is subject to t_1, where t_1 is the time, so that $x.\bar{x}.t_1 = 0$, then since in general

$$(x+\bar{x}).(x+\bar{x}).(t_1+\bar{t}_1) = 1,$$

we have $$(x+\bar{x}).t_1+(x^2+x.\bar{x}+\bar{x}^2).\bar{t}_1 = 1. \quad (4.18)$$

Under the condition t_1, $x.\bar{x} = 0$, similarly as regards space s_1. And likewise as regards all other conditions y, z, w, etc. in which $x.\bar{x} = 0$. Hence $t_1.s_1.x.y.z.w \ldots = A$, where A corresponds to the Absolute in which no contradiction is possible.

The recognition of this principle (by symbolic methods) serves to elucidate matters considerably, there being two classes of ideas dis-Continuous and Continuous. The Continuous element is due to the presence of the imaginary $\sqrt{-1}$, which becomes real on emergence. It is seen that an investigation of all problems is thus possible while adhering to the recognized laws of thought in

ordinary logic, i.e. 'static' as distinct from 'dynamic' thought or vice versa. The system which may appear somewhat pedantic at first sight, and as savouring of the almost obsolete methods of Scholasticism, will be found to justify its existence and its application, by the guidance afforded through the intricate labyrinth of thought in its higher spheres of transcendental ratiocination, and the fruits that its power of grappling with these most intricate questions yields, far exceeding indeed the limits of the ordinary human intellect, in range, in apprehension, and in control of the minutest detail, as well as in the treatment of the whole problem in its various aspects.

Quantitative relations form a particular case of the more general principles thus laid down, and physics is in truth but a branch of the more general science of logic, though physics gives but one aspect of reality. To group so many of the fundamental facts of Nature together in a single formula as this method enables us to do, yielding results such as Mendel's law and its relation to the law of sex through the transformation of groups of classes by still simpler laws, quite in accordance with the laws of thought, is an indication that the path we have chosen is destined to lead to still richer fields of untold wealth.

§ 3

Mendelian discontinuity in one scale of being is but the result of Darwinian continuity in another and a smaller scale. Thought aggregates in classes and leaves apparent gaps of insurmountable contradiction, which only a deeper conception of reality enables us to perceive as the continuous relationship of ideas in an ever interconnected series of classes of subjects and predicates in an infinite series.

The idea of a *quantum* in physics does not form an exception to this rule, so long as it is not regarded as the final unit, but merely as a halting step in the process of endless progression in the scale of infinite being.

The emission of an electron, or even a quantum from an atom, is an apparently discontinuous process, and constitutes a complete change in the whole of reality, like the omission of one of our terms in the logical series we have been considering. The continuity must be assumed to be in the infinitely small, unless we are actually dealing with reality. Is the quantum the noumenon, or is it still a phenomenon? The phenomena of life and mind would seem to indicate that in dealing with the quantum we are still far removed from the absolute reality. The noumenon may

in fact not be in space at all, and as the 'negative' of the phenomenon in the sense in which we have employed the term, it may well indeed be regarded as in another dimension from that of space and ordinary thought. This may be considered a hard saying. But moving along Hegelian and the analytical lines of the mathematical calculus, as the roots of the equation $x^n - 1 = 0$ attest, thought is n-dimensional.

Every atom, every cell, or class of units, x, y, z, w, p, q, r, etc. may be represented by the series

$$(((((((x+\bar{x})y+\bar{x}\bar{y})z+\bar{x}\bar{y}\bar{z}).w+\bar{x}\bar{y}\bar{z}\bar{w})p+\bar{x}\bar{y}\bar{z}\bar{w}\bar{p})q+\bar{x}\bar{y}\bar{z}\bar{w}\bar{p}\bar{q})r + \bar{x}\bar{y}\bar{z}\bar{w}\bar{p}\bar{q}\bar{r}) = 1, \tag{4.19}$$

this being the result of the telescopic relationships,

$$x.\bar{y} = 0,\ y.\bar{z} = 0,\ z.\bar{w} = 0,\ w.\bar{p} = 0,\ p.\bar{q} = 0,\ q.\bar{r} = 0^*$$

or that, x is y, y is z, z is w, w is p, p is q, q is r.

The Bohr atom is thus a particular case of the transference of a unit from one 'sphere' class or group to another. The change is brought about within as well as without the atom or group, and this produces the change of connotation and denotation, both in space and time. This change affects the rest of creation, but the message and the fact it entails take time in which to be propagated. The actual speed in material phenomena depends upon the precise nature of the equations, and is a matter for experiment.

In all judgements the categories of quantity, quality, relation, and modality form a necessary factor in the expression of the equation of the universe of experience and discourse. Space and time likewise invariably enter into the propositions, expressed or understood, as 'forms of intuition'. The actual physical relationships in space and time, as well as those of quantity, quality, and substance, etc., need not concern us here, they are dealt with elsewhere; we are at the moment concerned merely with the logical skeleton of the subject.

When the nucleus of a cell subdivides according to the principle just laid down for an idea or an event in which the connotation is altered, the denotation is increased in space in the manner indicated, then each portion may develop out in time according to definite laws. The displacement may be compared to that produced by double refraction of light through a biaxial crystal, or to the phenomena of karyokinesis imitated in this way (see Stephan Leduc's *Mechanism of Life*). The interference of light as manifested by two waves of the same wave-length moving from two

* Or $x\bar{y} = 0$; $xy\bar{z} = 0$; $xyz\bar{w} = 0$; etc.

slightly different points will give rise to interference bands, and, as Stephan Leduc has so admirably illustrated, may be imitated with rhythmic depositions of precipitates.

§ 4

Kurt Lewin (loc. cit.) has discussed the question of identity in time, of physical and biological objects in relation to the theory of evolution. This was previous to the application of relativity to such problems as are here considered. The reader will, however, find it instructive to peruse that work, and Windelband's *Über Gleichheit und Identität* (1910), in connexion with the problem of identity as presented in this book. For although the difference in our views is well marked, those contributions to the subject are of considerable importance. Lewin's notation will appear peculiar to the mathematical mind, and at first sight perhaps somewhat faulty and misleading. For instance he uses the expression $x^p = y$ to represent the conception of 'Genidentität' or existential relationship of two entities arising separately without any implication of 'power' p in the algebraic sense, and merely as a symbol like $x_p = y$. And $x^p \equiv y$, not as an identity in the algebraic sense, but as an identity and apparent continuity of mass and other such physical qualities, as distinct from the identity of the object itself in time.

The logical or Hegelian degrees of identity, or 'Momente', are not at all considered, and it is here that we part company from him, and follow the earlier mode of thought. Shifting the identity from the whole to the parts, as he does, hardly gets over the real difficulties in the problem. When I say that $x^p = a$, and $y^q = b$, and that if $x^p = y^p$, $\therefore a = b$, it does not follow that $x = y$, because only a particular degree or 'moment' p of x is identical with some other degree of 'moment' q of y, so that in general $x \neq y$.

If $x^p = v_1 x$, and $x^q = v_2 x$, then the algebraic analogy holds that $p = \log_x v_1 x$, and $q = \log_x v_2 x$, so that $p+q = \log_x v_1 v_2 x$, and the 'sum' is equivalent to the combination of both. This applies most appropriately to the theory of heredity that I have put forward. If $x^p = y^p = v_1 x = v_2 y = e^r$, where e is the common binding link between them, regarding e^r as the substratum or 'ideal' interval connecting the ideas of the two classes x and y, then using the trigonometrical analogy, which it is quite legitimate so to do, we obtain the following simple but general relationships:

$$r = p \log_e x = q \log_e y = \log_e v_1 x = \log_e v_2 y,$$

the moments or degrees of identity p and q, correspond to the qualifications v_1 and v_2.

Qualities as combined with a member of a class may be regarded as equivalent to a certain degree of identity of the particular member of the class, whilst in every case there is a particular degree of identity of the common idea e (corresponding to consciousness) which unites and comprehends the two in a mutual expression of identity. This conception of identity in difference is not brought out by the other method.

Not only in logic, but in biology as well as physics is the persistent identity preserved, notwithstanding the transformation from one class x to another y. The change from v_1x to v_2y is preserved in the degree of identity of e as represented by e^r.

Every possible idea is then represented by some particular degree of identity of e, which may be considered as the Ego, or monad.

Moreover, this can be expanded as we know from Boole's calculus, either by Taylor's theorem, or, as we have seen, by Fourier's series, so that other monads might respond to it through their appropriate characteristic partials. The Hegelian notion, then, which differs so much at first sight from the Leibnitzian, is seen to fall into full accordance with it; and conflicting ideas to respond as harmonious scintillations in an ever self-adjusting process. The sifting out of ideas, their selection, transformation and reform to types of unexpected marvel, present a united comprehensiveness that some apparent underlying idea or purpose, none other than the ideal process itself of mathematical expansion, unfolds through the logical development of the Idea.

Each monad, however, may have its own particular 'base', so that its degree of identity for any particular idea or notion may differ from that of any other monad, as it does in time and accordingly in space. Every idea then admits of co-ordination in a definite, similar, but different manner in each particular monad. Each monad has thus its own scale or standard, and no two are or need be alike. It is, however, conceivable that any two monads or ideas should coincide in some particular manner according to their moment or degree. The new wave mechanics comes nearer to this conception of reality than any previous system of physical thought. It comes indeed quite near the logical mode of thought that I have here proposed. For it is in fact but a particular case of the exponential expression of the idea in a logical proposition. The expansion of a co-ordinate q by Fourier's series as $\sum e^{2\pi int\nu}$, which has been admirably evolved by the use of matrices (see Max Born's *Probleme der Atomdynamik* (1926)), helps the idea of logical development just as if we were actually dealing with physical entities in the wave mechanics.

CHAPTER XVI

ON THE NATURE OF SUBSTANCE

§ 1

'WHAT do we know of the physical world?' asked Ostwald. 'Obviously only what is vouchsafed us through our organs of sense. But under what conditions are these organs set in action? Turn the matter as we may, the only principle we find common to all is this: *The organs of perception react in response to differences of energy between them and their surroundings.* Differences of energy constitute will force. In a world in which the temperature was everywhere that of our bodies we should have no experience of heat, just as we have no perception of the constant atmospheric pressure under which we live. . . . Since our knowledge of the outer world consists in its energy relations, what right have we to assume in this very outer world the existence of something of which we have had no experience? But energy, it has been urged, is something thought of, an abstraction, while matter is a reality. Exactly the reverse, I reply. Matter is a thing of thought which we have constructed for ourselves (rather imperfectly) in order to express that which is lasting in the changeableness of phenomena. Now that we begin to grasp that the actual in that which *acts* upon us is energy alone, we must inquire into what relation the two conceptions stand to one another, and the result is undoubtedly that the predicate of reality can be affirmed of energy alone.'*

This energetic theory has the advantage over the will theory in so far as it attempts to account for the potential as well as the actual. But energy after all is merely an *invariant* like matter, although it may be of a more general type. An *invariant* may be described as follows:

In our investigations of natural phenomena we group facts together until a general conception is found to be the connecting link between them, and that which remains unaltered when the constituent factors are made to vary is called an *invariant*, the determination of which results in the foundation of a general law.

Yet be it borne in mind that the statement that *energy* is nothing more than an *invariant* so defined, like matter, is, strictly speaking, based upon an argument not unlike that which Berkeley employed to prove that all our knowledge of the world is perception. Yet will and force or change of motion are but a kind of perception which Schopenhauer regarded as the parent cause of all other perceptions.

* *Science Progress*, 1896.

All we know of matter, says Ostwald, is that it is 'a group of various forms of energy co-ordinated in space', and the idea of mass is also such an invariant. Matter and energy, as the theory of relativity requires, being one, it may be maintained that the invariant in either case is only a relationship of ideas or perceptions. The phenomena of the so-called external world are merely manifestations of mind, and the relationship of certain ideas or perceptions in our own, as well as other, minds. The relationship which expresses the law of the 'conservation of energy' may be compared to certain relationships which exist in the theory of numbers. Standing as they do for those ideas which represent to us the outer world, we may be led to regard that permanent relationship as representing, symbolically at least, a substance which we may call energy. But there is no more reason for believing in this substance than in matter, which is also represented by somewhat similar constant relationships.

Thus, energy is no more a substance than matter or electricity. When divested of all primitive notions it is found to be as unsubstantial as anything else; to be, in truth, a particular mode or property of mind; a relationship of ideas or perceptions. To the untutored intelligence, gross matter is the ultimate substance. To the scientist it is the proton and the electron or a group of these; a superposition of waves. To the idealist it matters not in the least which it is. If it is possible it must be perceptible; if it is a reality it must be perceived.

'Properly and immediately,' says Berkeley, 'nothing can be perceived but ideas. All material things, therefore, are in themselves insensible, and to be perceived only by our ideas. And it is most evident that *no idea can exist without the mind*. Besides spirits, all that we know or conceive are our ideas. It is evident that the things I perceive are my own ideas, and that no idea can exist unless it be in a mind; nor is it less plain that these ideas or things by me perceived, either themselves or their archetypes, exist independently of *my* mind. . . . The things I say, immediately perceived, are ideas, or sensations, call them which you will. But how can any idea or sensation exist in, or be produced by, anything but a mind or spirit?'

Unlike some idealists of our time and contrary to popular belief, Berkeley did actually believe in the existence of a world independently of ourselves, although that world consists merely of ideas: those ideas being perceptions caused in us by the Absolute Divine Mind.

Beautifully does he express it, and the reader will do well to refer to the charming *Dialogues between Hylas and Philonous* to acquire a grasp of his conceptions. Our immediate object is to

drive home to the reader's mind the idea that this potential will power in Nature, call it the external world if we wish, or the divine mind with Berkeley, it is but a system of ideas; that which rightly speaking IS.

The following passage from the *Dialogues* substantiates the interpretation here given of Berkeley's view, so often misrepresented by those who are only acquainted with his works at second-hand:*

'Look! Are not the fields covered with a delightful verdure? Is there not something in the woods and groves, in the rivers and clear springs, that soothes, that delights, that transports the soul? At the prospect of the wide and deep ocean, or some huge mountain whose top is lost in the clouds, or of an old, gloomy forest, are not our minds filled with horror?

'Even in rocks and deserts is there not an agreeable wildness? How sincere a pleasure is it to behold the natural beauties of the earth! To preserve and renew our relish for them, is not the veil of night alternately drawn over her face, and doth she not change her dress with the seasons? How aptly are the elements disposed! What variety and use in the meanest productions of nature! What delicacy, what beauty, what contrivance, in animal and vegetable bodies!

'Raise now your thoughts from this ball of earth to all those glorious luminaries that adorn the high arch of heaven. The motion and situation of the planets, are they not admirable for use and order? Were these (miscalled *erratic*) globes once known to stray in their repeated journeys through the pathless void? Do they not measure areas round the sun ever proportional to the times? So fixed, so immutable are the laws by which the unseen Author of Nature actuates the Universe. How vivid and radiant is the lustre of the fixed stars! How magnificent and rich the negligent profusion with which they appear to be scattered throughout the whole azure vault! Yet if you take the telescope it brings into your sight a new host of stars that escape the naked eye. Here they seem contiguous and minute, but to a nearer view immense orbs of light at various distances, far sunk in the abyss of space. Now you must call imagination to your aid. The feeble, narrow sense cannot descry innumerable worlds revolving round the central fires; and in those worlds the *energy* [the italics are mine] of an all-perfect Mind displayed in endless forms. But neither sense nor imagination is big enough to comprehend the boundless extent, with all its glittering furniture. Though the labouring mind exert and strain each power to its innermost reach there still stands out ungrasped a surplus age immeasurable. Yet all the vast bodies that compose this mighty frame, how distant and remote soever, are by some secret mechanism, some divine art and force, linked in a mutual dependence and intercourse with each other, even with the earth, which was almost slipt from my thoughts and lost in the crowd of worlds. . . . What treatment, then, do those philosophers deserve who would deprive these noble and delightful scenes of all *reality*?'

* I directed particular attention to this passage and its bearing on the correct interpretation of Berkeley's philosophy, in an article in the *Oxford and Cambridge Review*, Michaelmas, 1908. Sir James Jeans in his Rede Lecture on *The Mysterious Universe* directs attention to Berkeley's teaching.

Who can doubt from this that Berkeley believed in a world external to our own minds, although he understood that world to be a world of perceptions in some other mind? Indeed the crux of the whole subject is that mind and matter have reciprocal relations, but that this relationship is between various objects of mind.

Mind constitutes the very essence of matter, and not of matter alone but of all reality. It is the Alpha and the Omega of all existence, for 'to exist is to be perceived'. And to ignore the fact is to suppose, as Herbert Spencer is thought to have done, that an underlying substance subsists, that substance being unknowable. Herbert Spencer surely meant that it was unknowable to us and all human intelligence, but not necessarily unknowable to an omniscient being, else he would have meant nonsense; and it is not likely that he was so misled as to interpret his unknowable thus: although critics and opponents, who are many, have given this interpretation to his words.

To suppose then that all this vast reality which we call the universe is dependent for its existence on ourselves, would be to accept something approaching, if not very much worse than, the Ptolemaic system of the world. Berkeley's theory was that these *permanent* perceptions of a so-called outer world are produced by the Absolute Mind, by God Himself on our minds, and that they exist all the while in the Divine Mind. In this latter respect, contrary to the popular belief, he was perhaps in accord with Plato. There is, however, a profound difference in their teaching.

What has been said as regards the external world directly perceptible to the senses, applies even more effectively to the inner constitution of matter postulated by modern science such as the atomic theory: the theory of electrons and of the luminiferous aether. It is clear that these assumptions, and even the commonsense assumption of matter itself, are of use merely as instruments which facilitate research: as a convenience for the solution of practical questions which, like quadratic equations, admit of two solutions, or equations of the nth degree of n solutions. As a halting-stage they are most assuredly something corresponding to atoms and molecules, although not the ultimate reality; not more so indeed than gross matter itself. The relationships do not prevent each system from fitting one into the other. And although it may appear at first sight to have the appearance of eclecticism to regard this matter thus, I do not see that when placed in its proper light each system does not in some respect represent an aspect of the truth. Only when pushed to its strictly logical extreme does pure idealism hold its own.

To Leibnitz, on the other hand, the monad was an atom or centre of force of infinitesimally small magnitude, possessing consciousness in some dim remote degree. It was in fact, not the atom of the chemist, but the ultimate unit of substance; a conscious unit too. The relationship between these systems and that of pure idealism seems to me worthy of attention as a means of reducing Leibnitz's 'dynamism' and Ostwald's 'energeticism' and Schopenhauer's pragmatism to strictly idealistic conceptions. It is indeed a most interesting task; for the idealism of Leibnitz, of Schopenhauer, and Berkeley are then entirely different systems of thought, as Hegel would have maintained. And yet if they do actually represent but different aspects of the truth, a mode of reconciliation between them should be possible. They must not be absolutely contradictory, yet one may be a deeper philosophy than the other. Leibnitz's view fits in as I think with all that science has to teach, whilst Berkeley gives us the ultimate interpretation of it all. It is not concerned with where science may begin or end. Only perceptions concern it; and only of perceptions does anything that can occupy our attention actually consist. In its ultimate aspect then, it is of no very great concern whether the universe can be resolved into atoms and monads, so long as these are admitted to be mere perceptions or ideas, as much so as the chair I sit on, the table on which I write, or the pen I hold, which constitutes an aggregate of them.

Leibnitz's monads are as capable of admission into this sacred system as the vulgar notion of matter or materialism. It is a quality of experience that the world may be pictured as a system of atoms, and doubtless ultimately of monads; and if these are elements of mind-stuff a particular aggregation of them will constitute the living organism. The unity which constitutes consciousness points to the fact that there is one particular monad that regulates the rest, as a general regulates an army. Circumstances favour this signal triumph and it is idle to speculate as to whence or how it may arise; for there is no more reason to suppose that monads are all alike than that organisms in our own scale of being are all alike.

Far into infinitesimals of the first, of the second, or the third orders, nay, into infinitesimals of the infinite order, lies the realization of the monad. To suppose the contrary is contrary to what common sense, or preferably higher sense requires. Hence out of millions of millions of millions of monads one type should possess such qualities; one that in countless ages should manage to evolve in certain groups or aggregates of grosser matter, as the

fittest to survive and to hand down its type in kind through a series of countless generations; for of time there is no limit, and of varieties of monads and their countless combinations in time no end.

Idealism indeed does not after all affect the question, although idealism be true; for these monads are self-conscious reflective ideas. That appears to be their essence. But of this more fully presently and anon.

A monad is a centre of force or will power; but it is also conscious. Moreover, it is as much an infinitesimal as a point can be said to be so.

The host of monads does not alter in the very slightest the fact that, individually and collectively, each and every one of these constitute a system of ideas in that vast and comprehensive mind which embraces all; conceiving Nature in her totality. The monad, of whatever kind, though a unit, is a world in itself; a faint reflection of that intelligence which apprehends the whole; a self-conscious idea inspired and animated by the Spirit of the Divine.

James Ward says, 'I should not like to pin my faith to Leibnitz, but of all the dogmatic philosophies his seems to me—in one feature at any rate—by far the best,'* and McTaggart† seems to regard the question in a manner which a consistent Leibnitzian would not, in part at any rate, reject. McTaggart would look upon all monads as totally distinct—since presumably he would deny the continuity of infinitesimal quantities. To put it vulgarly, but truthfully, the proof of the pudding is in the eating, and we know that mind does act on mind, and hence the monads cannot be independent, so that there must be a continuity between them. Such a continuity of individual minds would only be unified in a single omniscient and universal mind. The Dedekind point is distinct from a line of which it is a part, but the infinity of points constitutes a line.

There is no substance but mind. And in that substance it will be found that a distinct continuity prevails in all its countless forms. In its various self-conscious ideas or monads there are points of greater or lesser concentration. But throughout a unity and plurality is evidenced at every point. Telescopically, as it were, each point is a unity; and microscopically a microcosmos, carried to an infinitesimal of the *n*th or infinite order or degree. The sum-total, however, is a unity, as much so as the individual parts, there being an apparent discontinuity amongst aggregate parts, and an actual

* *Naturalism and Agnosticism*, p. 181. † *Hegelian Cosmology.*

continuity when the minutest are examined in their minutest detail; for in this continuity alone does the continuity of the whole become possible.

§ 2

An illustration may be given in the case of the conception of motion. When minutely analysed it is found to be unintelligible. But we know that it is possible, as all experience proves. Ergo it is only intelligible to a higher intelligence than ours. The idea of motion in itself is apparently self-contradictory, as Zeno showed, since at any particular instant a body must occupy a particular place, and at the next instant a consecutive position in space. It cannot both be and not be at a particular place at a particular time. But its position is continually changing. Is there not an apparent contradiction here, which seems to baffle the ordinary canons of pure reason? The difficulty indeed does not transcend the human intellect, and lies in the conception of time which involves a one-one relationship between every point in space and every corresponding instant in time (see p. 21).

My object has been rather to emphasize the manner in which the monadology helps to elucidate the difficulties that lie in the way of a reconciliation between pure idealism on the one hand and gross materialism on the other; a buffer as it were between Schopenhauer and Berkeley.

We have seen that Ostwald's Energeticism destroys matter and reduces everything to energy, centred as it is in points in space. We find, however, that when carried further these centres of energy are centres of mind-stuff or ideas potentially or actually self-conscious, and, in the limit, ideas in that Mind which conceives and animates it all. These self-conscious units or spirits are realities apparently independent of each other, but they are parts of, related to, and connected with the whole, like points in space, though not necessarily as ordered.

We have in fact reached approximately at least the stage attained by Lotze.

This view of the problem of mind and matter appears to afford the most promising lines for investigation.

It is clear that the evolution of life and mind from so-called inorganic matter in the remote past may be reconciled with the theory that physical and chemical atoms are systems of such monads. In the atoms of the same elements, then, there would obviously be variations which do not affect their ordinary physical

and chemical properties as aggregates. Amongst these 'atoms' in their various elements and compounds there must be a certain amount of natural selection, by the survival of stable forms or groups, or sifting of those types which can manifest and propagate in an efficient manner the characteristic properties of life.

A single monad out of millions would then be the chosen one that controls the development, after passing through countless inorganic and organic forms into the complex types associated with the living organisms of to-day. And the vital principle in organic and inorganic matter is to be sought in such variation in the atoms of the organism.

The manner in which such variations may arise and be followed up is discussed in the sequel, i.e. the chapter dealing with the evolution of nebular and spiral atoms in the various stages of condensation from the primordial substance of mind-stuff to the structure of highly-developed mind.

CHAPTER XVII

THE EVOLUTION OF LIFE

THE study of life's nature and origin comes within those departments of science which deal with the invisible and ultra-microscopic forms of matter, and, as I have tried to indicate elsewhere, paradoxical as it may seem within the realm of chemistry, or indeed more appropriately physics, than within that of botany or zoology. It is, perhaps, as remotely connected with these as these are with human physiology; nay, even with psychology. For the problem of life's origin is one that resolves itself into the gradation of the infinitely small, and is ultimately lost in it, a fact which concerns the physics far more than it does the biology of to-day.

Viewing it as we may, it leads us, as I think, to infer the atomic constitution of vital substance—vital atoms, monads, entities as separate and distinct, and perhaps immutable, as the atoms of the chemist, or the electrons of the physicist, by evolution in infinite time.

No prejudice or predilections, as to what does or does not come within the sphere of any particular science, should be allowed to stand in the way of developments on these lines; if there are reasons which indicate that the recognized limitations should be extended or removed.

Now in dealing with vital phenomena, from a physical point of view, one fact stares us in the face each time we approach the problem; and that is, the optical activity, or asymmetrical structure of living matter; and the fact that such asymmetric structure can be naturally obtained, through the agency of its vital processes.

This is a phenomenon which undoubtedly runs parallel to and is concomitant with the phenomenon of life itself.*

How did the asymmetric structure arise? Is its asymmetric property that which gives the unstable but directive quality to living matter? Is it, and is living matter with it, as persistent, as approximately stable, as the atoms themselves? How are physical principles in the phenomenon of *biotic* energy or potential life-activity to be explained?

The argument, from continuity alone, might lead us to expect that the electron is only a halting step in the great scale of being; and that, whether we should ever succeed in demonstrating it or

* S. Smiles, Sir William Pope, and others have obtained optical activity in sulphur, selenium, tin, and silicon, when these behave as tetrads.

not, it, too, in its turn will be found to be an aggregate of smaller units or quanta of energy. So on *ad infinitum*. There is, however, no actual demonstration of this possible at present, and quanta suggest a limit. Wave-mechanics throws new light upon the subject.

Now, my endeavour has been to show that biophores, the ultimate nuclei or vital units in their final state, are aggregates of smaller units. There is a stage through which, like spiral nebulae and dwarf stars, on a totally different scale, they would pass before being transformed or condensed into stable systems, like chemical atoms.

As all nebulae are apparently resolvable into these spiral forms, so the condensing aggregates of electrons not less than those of matter in its grosser states would have to assume the peculiar shapes or curves, which characterize their larger analogues.

We must not expect that we shall ever reach the smallest any more than the largest of things; for in the mathematical and physical there is nothing either great or small, but thinking makes it so.

The problem of life transcends physics because for many reasons it not merely depends upon the ultra-microscopic, but in turn upon the atomic, ultra-corpuscular, or ultra-electronic, *ad infinitum*.

The electron or corpuscle should be an aggregate, and not an indivisible unit. On this basis, the formation of such nebulae and spiral atoms may be conceived. But the spiral forms may be composed of waves. The actual corpuscles which form the planets of the atomic solar systems would seem to be in their turn but aggregates of smaller units.

My chief reasons for holding this view are as follows:

As Clerk Maxwell had pointed out (see p. 370), the variety of phenomena of life could not be accounted for by the small number of atoms in the organic cell. The chemical atoms, indeed, of the same element may be regarded as resembling each other only in certain respects, as one individual resembles another. But if they are aggregates of smaller units, as I have tried to emphasize in *The Origin of Life*, there would be opportunities for variations in atoms, which, as chemical units, are the same. The atoms of the same substance would, so to speak, have merely a family likeness, and there are strong reasons for supposing that the phenomena of life are thus dependent upon the ultra-atomic constitution of matter. A process analogous to natural selection or survival of stable groups would ensue.*

* See my article on 'The Evolution of Life, or Natural Selection in Inorganic Matter', *Monist*, April 1908. The discovery of Isotopes by F. W. Aston constituted one of the most remarkable achievements of our time.

Now the general physical properties of gross matter are explicable, as we know, on the supposition that there are small particles called *molecules*; the chemical properties on the supposition that there are smaller ones called *atoms*; the electrical properties on the assumption that there are still smaller ones *protons* and *electrons*, these resolving ultimately into quanta of energy and photons. The biological properties can, I think, be explained on the assumption that there are yet smaller units—since, as we shall see, such phenomena demand it—of which biophores or biogen are nebular forms. If there are biophores with an asymmetric structure, that asymmetric structure and all its varied properties can be explained on the supposition that the electrons are aggregates of smaller units; an assumption which other phenomena demand. There is nothing unphilosophic in this hypothesis. And if the atomic theory is to hold in its application to biological, as well as to purely chemical and physical phenomena, the facts of the case demand it. We cannot frame the nebula analogy for electronic systems without postulating that the electrons or corpuscles are aggregates of smaller things. The electron may be compared to a planet; the nuclear positive ion or proton to a sun;* but if these systems have been formed by condensation, they must be the condensation of still smaller things.

Germ-plasm, that is, the biophoridae, would thus consist of a nebula of uncondensed matter, of corpuscles or electrons in the state of formation, as planets and solar systems are evolved from atoms. This is not an absurdity, as may at first sight be imagined, but merely an extension, and a logical extension, as I venture to think, of the theory that atoms are such miniature planetary systems. Wave-mechanics, by the superposition of waves, yields similar results, as the group waves can be as complicated as we please to make them.

The atoms and electrons may be regarded as the condensed products of the primordial substance, which I have called *biogen*, and a network of cells may be formed by the same process.

Sir Arthur Schuster has written in the following terms in discussing this theory of biogen in its relation to the origin of life:†

'The discoveries of the last few years have in one respect inverted our previous ideas of the problem of creation. That there was only one substance, and that all matter was ultimately derived from that substance, has,

* So far as I know the planetary theory of atoms and electrons was proposed by Sir Joseph Larmor in his 'Theory of Electrons', *Phil. Trans.*, 1896. But the Rutherford atom is a more modern conception based upon later ideas relating to radioactivity which has proved so fruitful in this respect.

† *Manchester Guardian*, 27 April 1906.

I believe, been always a tacitly accepted axiom of scientific thought. But we look upon the more complicated structures as having been gradually built up from the simpler ones. Hydrogen being the body which has the lightest atoms, and other bodies having atomic weights which are approximately multiples of that of hydrogen, the thought was natural that hydrogen was, if not the ultimate atom, yet the body which formed the bricks out of which other so-called elements were built up. At last, however, we are able to watch the actual formation of one element out of another, but instead of the more complex bodies being formed out of the simpler ones, the reverse is the case, and one of the lightest elements, helium, is evolved from one of the heaviest elements, radium.

'Where we can follow the process of evolution, it is therefore the complex body which, through instability, yields the simpler one. Scientific men love to generalize, and it is therefore natural that they should quickly have inverted their former ideas and have begun to look at the most complicated structures as the ones which were formed first. If this idea is driven to its extreme—though not necessarily its logical—limits, we should be led to the conclusion that living organism is the primordial element, and that inanimate matter is only the result of the decay of living matter. This I gather to be Mr. Burke's view. He compares, indeed, the derivation of ordinary matter from the original bio-element with the evolution of the solar system, out of the historic, though now somewhat discredited nebula. It is undoubtedly a new and an original view, and one that must be taken seriously.'

I owe to Sir Arthur Schuster a deep debt of gratitude for thus emphasizing what at the time many of my friends could not be made to appreciate. Some investigators are as absorbed in their own special work as they are indifferent to its relation to anything else. I say this in all good faith, appreciating as I do the high value many investigators' opinions in their own special, but narrow, line of work may have. But the great discouragement they can mete out to any scheme which does not fit in with that which they themselves may happen to be engaged upon, is not always stimulating or satisfactory. New ideas, no doubt, require to be familiarized. If they do not at first seem to be quite orthodox—as new ideas seldom are—or if they are not altogether on similar lines to those on which others have been working, the promulgator thereof may expect to have many bricks, like books and candlesticks of old, flung at his wearied and bewildered head.*

Quite apart, then, from the question as to whether the views I have ventured to put forward are orthodox or heterodox, let us consider them in their relation to the great problem which not

* Sir James Jeans assumes that atoms of higher atomic weight than those known here below exist in the stars, and thus accounts for giant and white dwarf stars, supposing that they disappear in the form of radiation. The theory here put forward and first advanced in 1906 presupposes nebular atoms of what we should perhaps now call photons, but requires atoms of higher atomic weight than those known at present.

merely biology, but physics now presents; and the optical asymmetric structure of living matter and the concomitant or inseparable connexion between living matter and asymmetric structure. For as living matter has never been produced from dead matter, as familiarly understood, so polarizing matter always accompanies that which is living or has lived, though as we have seen it may exist without it.

As we say, if the atomic and electronic systems have been formed by a process of condensation of a primordial substance, the transitory stage of spiral nebulae is to be expected, the probability waves corresponding to nebulae of photons. But it may take many thousand years to pass through this transitory stage; even as the astronomical nebulae on the larger scale take many million million years to be condensed into a solar system.

In such circumstances, then, the atomic nebulae may, to all intents and purposes, be regarded as entities, as much so as the individual atoms; unstable as these may really be. Their spiral nature, however, would in certain circumstances rotate the plane of polarization of light; as, for instance, when the atoms are all right-handed or left-handed.

Now such spiral atoms or groups of probability waves, if they may be so called, will in the course of time be converted into more stable atomic systems, becoming radioactive, and ultimately inert, the latter being the lightest elements.

The lighter elements being more mobile admit of a larger number of combinations, and hence it is a curious circumstance that the principal constituents of protoplasm are carbon, nitrogen, oxygen, and hydrogen, the lightest of each group; and that silicon, phosphorus, sulphur, chlorine, &c., the second lightest of the respective groups, come next, but in far lesser proportions.

This point is most suggestive, indeed. And the formation of cellular protoplasm from such atomic systems is rendered probable enough when we consider the development of mimic or artificial cells in protoplasm by the action of radium salts upon it. Experiments made with gelatine, not containing albumin or protoplasm, yield negative results. They do not in any way touch the point in the argument. (See *Origin of Life*, p. 123 and pp. 173–4.)*†

The complexity even in the nucleus itself would be very great

* See also Kuckuck (loc. cit.), p. 28.

† I have had stored many tubes sterilized at Cambridge in 1905–6, some of which may be opened at regular intervals of say fifty years. If the process is slow it would be very slow, and the experiment is obviously not for a day, or a nine-days' wonder.

indeed, so much so that to account for its great varieties and variations presents insuperable difficulties far exceeding anything that unaided imagination enables us to picture to ourselves. (See *Origin of Life*, p. 224.)

This is the inference which analogy, extrapolation, and experimental results may lead us to draw. I have said elsewhere that

'We may regard the ultimate vital substance which seems to play so mysterious a part in the phenomena of Nature as corresponding to a state of matter which should be described as neither molecular nor atomic, but of matter in the process of becoming—a state of electronic, or really of biogenetic, aggregation, from which matter in the course of time has been evolved, the hiatus, the borderland or critical state, between disintegrated and integrated electricity.

'Biogen may be regarded as the intermediate state between free electricity and condensed electricity, which we call matter—the hiatus between electricity as we know it and matter as we know it. It may be looked upon as the nuclear substance—in its ultimate form, as the material in which the energy and potentialities of the cell are stored, and from which they afterwards emanate. It possesses those properties which enable it in suitable environments to produce cells like the artificial ones we have demonstrated. But it possesses also the power of exciting vigorous metabolism, and of developing into more diverse forms.*

'This nuclear substance, consisting of biogen or the ill-defined bio-elements, is what I regard as the basis of the manifold properties of living things we see around us.'

This view of the phenomena of life shows us that it is merely one phase in the never-ending series of gradations.

Sir James Jeans' and Sir Arthur Eddington's theories of dwarf stars and spiral nebulae have thus their analogues in the microscopic worlds.

The more firmly we can rivet the idea in our minds that our place in the scale of being, like our nature, is a middle nature, and that dimension in space is purely relative, the more ready shall we be to expect that phenomena, not unlike those in our own sphere and yet different, should take place on a smaller scale in the infinitely smaller worlds than ours.

Now I think this view cannot be too strongly emphasized. And I would discourage those who look for an explanation of the vast complexity of the phenomena of life we see around us, and of the

* The metabolism may be set up and maintained by radiation from the nucleus. (See *Origin of Life*, Chapters XII to XVIII.)

The synthesis of formaldehyde by light could have been effected by γ-rays, in caves, or at the bottom of the sea, or underground. It is for this reason I think that radioactivity played a more important part than light in the synthesis of living matter. It is extremely unlikely that the ultra-violet light was sufficiently intense at the surface of the Earth.

still far greater complexity, that must have been eliminated by natural selection, in the mere atomic properties with which the chemist is familiar. I would try to emphasize once more the inadequacy of such chemical units to account for the facts which are presented to us; and to impress the importance which such smaller and smaller units we have deemed it necessary to postulate in the infinite gradation must exercise upon the phenomenon of life in our own scale of being.

Recent work, and in fact, all work, on spontaneous generation would seem to come under the category of uncertainties. It is difficult to see how it could be otherwise; and how it could be proved that by any process of sterilization in such experiments, the last traces of life have ever been annihilated. For although the organic life we are familiar with may be destroyed, some of the biophores or vital units of Weismann, ultra-microscopic forms, may, and most probably do, remain untouched or unaffected, like the immutable elements which compose the chemical constituents of the living protoplasmic substances themselves.

This everlasting controversy about that elusive and receding phantom called spontaneous generation, that disappears from our sight as soon as we may have thought we had grasped it, owes its persistency, no doubt, to the difference of perspective, if we may put it so, in which the real point at issue has been viewed by those who have most eagerly pursued it.

Whether we are to regard Weismann's biophores or vital units as forming a continuous scale of ever simplifying vital forms *ad infinitum*—an aspect of the question which seems to be the most reasonable—or whether we should regard life as appearing from absolutely inanimate matter through mere chemical processes, by the artificial production of proteids, as Professor Fisher, of Berlin, tried to do, remains the one point at issue in the great problem of the origin of life.

Dr. Charlton Bastian had for thirty years or more affirmed again and again, against his colleagues in the world of science, that such a phenomenon does take place every day around us, and may be seen in the microscope to-day. Whether he was right or not the future will decide. He appears to have stood alone in the belief in present-day spontaneous generation.*

All the phenomena he has observed may be explained, I think, and yet not lose any of their importance, but perhaps have some importance attached to them, on the supposition that they are the

* I have maintained all along that artificial life would not admit of being identified with actual life handed down in the evolutionary series.

results of the growth of biophores in their development from ultra-microscropic forms.

In this germ-plasm, or biophores, we must seek the mystery of the origin of life. Weismann does not profess to go further. He is content to postulate that all the vast complexity of development of which these biophores admit results from a process which is inherent in them. But in order that this might be so, it is necessary that the physicist should step in and point out that there are difficulties from his point of view. Atoms and electrons, he would say, do not suffice to explain the vast complexity of phenomena which these biophores require. And if the biologist were to press the point, as no doubt he would do, there is, I think, no alternative but to regard the atoms and electrons, which we have supposed to be the smallest, as in reality but aggregates of smaller things; whilst the assumption which is forced upon us affords, it is true, an opportunity to explain, not merely the evolution of the phenomena which are more properly biological, but the evolution of matter, electrical and chemical, from that self-same primordial substance.*

We are forced, I think, from the assistance which biology thus lends us, to recognize in our own work as physicists, phenomena which that work would scarcely have led us to postulate.

In this, as in some other instances, it may be perceived that biology and physics overlap; and that, as in the case of geology, we may get a hint from the biologist that there is *a residual phenomenon* which we, as physicists, have yet ignored.

Still, it may be urged in passing, that the optical asymmetric structure can be explained, and has been explained, without any such hypothesis. True, but the invariable connexion between it and living matter has not been. And in the *Origin of Life* my endeavour was to show that there is an element, a bio-element, perhaps a group or groups of such, possessing a vast store of potential properties, and biotic energy, which enable it to act as a nuclear substance, rotate the plane of polarization, and though last, not least, as an element admit of an almost indefinite variety of forms—as spiral nebulae do, thus giving rise to variations—and yet be, from the chemist's point of view, like other elements. It would merely be the stage through which nebular biogen would have to pass in being transformed into carbon. And to that almost indefinite variety of forms of biogen which the atoms would assume —or (should we say?) the various stages of condensation—may

* Part of this chapter appeared in my articles in *Knowledge*, March and April 1907.

be traced the indefinite potentialities of the biophoridae. That indefinite variety cannot be traced to their complex molecular structure, which is much too simple to be composed of many atoms. Therefore it is that we seek to account for that indefinite variety by something else—in the difference of structure of the atom, the atom of the nuclear substance.

Professor Japp has found in the origin and behaviour of the tartrates one of the strongest reasons for his predilection towards biogenesis. In his famous presidential address to Section B, at the British Association, he emphasizes how it was that physiologists had been carried away by Pasteur's later work, in which the biological element predominates; and how they neglected his earlier work. And yet he says:

'It ought to be of interest to physiologists, not merely because it is the root from which the later work springs, but because it furnishes, he is convinced, a reply to the most fundamental question that physiology can propose to itself; namely, whether the phenomena of life are explicable in terms of chemistry and physics; in other words, whether they are reducible to problems of the kinetics of atoms; or whether, on the contrary, there are certain residual phenomena, inexplicable by such means, pointing to the existence of the directive force which enters upon the scene with life itself, and which, whilst in no way violating the laws of the kinetics of atoms—though, indeed, acting through these laws—determines the course of their operation within the living organism.'

And, again, if we view the matter from a physical standpoint, and from that standpoint alone, it does not appear, as Japp insists, that the explanations so far offered are from a physical point of view quite satisfactory. The statement that the carbon atom has a twisted structure conveys little to the mind of the physicist, unless the conception can be expressed in terms of mechanics, i.e. traced to a dynamical foundation.

Now this is the substance of which germ-plasm, or biophores, consist. Weismann, in his *Evolution Theory*, translated by Sir Arthur Thomson, vol. ii, p. 369, says:

'In regard to them (biophores) alone is the possibility of origin through purely chemico-physical causes, without the co-operation of life already existing, admissible. It is only in regard to them that spontaneous generation is not inconceivable. We must, therefore, assume that at some time or other in the history of the earth, the conditions necessary to the development of these little invisible particles must have existed—they may exist to-day; and that the whole subsequent development of the organic world must have depended upon an aggregation of these biophores into larger complexes, and upon their differentiation within these complexes. We shall never be able, then, to observe spontaneous generation directly, for the simple reason that the smallest and lowest living particles which could arise through it, the biophoridae, are so extremely far below the

limits of visibility, that there is no hope of our ever being able to perceive them, even if we should succeed in producing them by spontaneous generation.' And I may add that they would be expected to be entirely different from organisms of the evolutionary series. (See pp. 5, 241, and 267.)

The late Sir Bertram Windle, in commenting upon this passage in the *Dublin Review*, April, 1906, says he does not believe in Weismann's continuity of views. He compares his assumption within assumption, as he puts it, to the parenthesis within parenthesis, and re-parenthesis, which Mark Twain complains of in the 'awful German language'.

He pathetically remarks:

> 'It is assumed that the substance within the germ-cell is not simple, but complex. It is assumed that this complex body is made up of determinants for different parts of the body. It is assumed that these again are built up of vital units, each living its own life; struggling with its neighbours, influenced by the nutrition by which it is bathed; and finally, by an all-embracing assumption, these unseen, unprovable vital units are erected into a new family of living things, the "biophoridae", and we are told they were spontaneously generated, and that no man can prove the contrary, for they are, and must always be, invisible.'

Dr. Windle was no doubt aware as one of the foremost scientific exponents of his time, that molecules, atoms, and electrons are equally invisible; and yet no man of science at the present day would question their existence as physical entities apart from the explanation of their nature as group waves in quantum mechanics. Visibility is not always the test of the reality of phenomena. If it were, the whole fabric of modern molecular physics would break down. It is the coherency of a theory, the fact that it can predict events which can afterwards be verified, that lends it value. Now, although I do not for a moment underrate the importance of Windle's argument—for I know well that it was the inadequacy of the germ-plasm hypothesis, in its present form, and not the invisibility of biophoridae, with which he was dissatisfied—I think that it is in this respect physics may move forward and lend a helping hand to the sister science of biology. When we come down to biophoridae, we find ourselves at length in the region of molecular dimensions; and in the realm not merely of the chemist, but of the physicist as well. It is indispensable that if the discussion is to be continued, we should form an *entente cordiale*, for the biologist is no longer in undisputed territory. His resentment at the interference from without in matters which he has been accustomed to regard as exclusively his own concern, now awakens in him the sense that at last he has reached a foreign land, where

the modes of thought and of expression are adequate in themselves, but not quite those with which he has been familiar. In this unconscious trespass the physicist, too, has much to learn from this welcome visitor. And it is that which it has been my object in this book to point out.

Yet, as Windle imagines Weismann to say, 'Give me my determinants, and I will explain anything you can ask me.' Certainly, and give me an evolutionary hypothesis of matter, such as that which I have sketched out, and I will show you that your determinants are a necessary consequence of it. There is nothing outrageous or improbable in this evolutionary conception of corpuscles and atoms from a primordial substance, whose elements are necessarily much smaller still. This is what the idea of continuity leads one to expect *ad infinitum*. The quantum of energy and the photon are now the smallest physical units as the atom and the electron were thirty years ago. We acquire, I think, a clearer and more comprehensive grasp of the world, by thus looking upon phenomena as the result of a continuous gradation of events from the infinitely great to the infinitely small.

I know well there are many who will not enter into this frame of mind, or swallow this idea, if I may put it so. But there are others who can, and do. The fascination for this great idea, as I say, ever presses upon me the importance of thinking of the farthest limit we may reach to be a halting step towards that which may be reached. We have no fixed zero or origin. And the mystery of life lies in that untravelled country which no investigator has traversed, or will ever reach—the world of the infinitely small. Leibnitz, as we have seen, regarded his monads and atoms as points, infinitely small, and without extension in space, but extended in another dimension in the world of mind. This is the view of Nature which, as I have tried to show, approaches or resembles somehow the theory of matter and mind-stuff which may be made to harmonize with our theory of life. In the monadology, and in the depth of intellect of that renowned philosopher's great conception, may we seek to grasp the meaning of that middle nature which is our being, in which we may most readily perceive the subjectivity of all scientific phenomena, and the basis of that consciousness which makes our knowledge possible. Biologists, with all due respect to them, will do well to recognize where the real difficulty lies if it is to be overcome.

The supposed purpose or design in living matter is the result of the self-assertiveness of certain stable groups of monads which by the struggle for existence and natural selection have acquired the

controlling power amongst certain groups of monads in the course of evolution; and as such have obtained a mastery amounting to a 'vital force'. The part played by chance, however, and the operation of physical processes must be regarded as the essential links in the chain of cause and effect though at the periods of development of living matter from the primordial substance whose units are monads.

As to the nature and behaviour of these, the preceding and following chapters afford considerable evidence, not only from the purely physical aspect but from the contact of experience with the grim realities of life.

A gradual evolution of the Idea is like the development in a landscape of a distant point as we approach it. The object here appears to grow from the infinitesimally small into the highly complex object of our direct experience; as the change in perspective makes the relative appearances of things alter.

The expansion of the view as it seems to emanate from a single dot in the field of vision presents in the case of life itself the mysterious appearance of the genii of the *Arabian Nights*. And our experience in time is like that of the view from the locomotive of a train, as object after object comes from the unknown, as it were, the infinitely small, into the range of vision, developing, growing, till it suddenly disappears. Such is Life. And such the ever-changing panorama or cinematographic vision that our passage through this world unfolds.

Far as we may go, and we profess to go into the infinitesimals, we do not profess to reach monads that are all alike; for in the limit they are, and must be different.

We may go to infinitesimals of the first order, or the second, or the third, and so on. There is no limit. Things unlike never are or can be made precisely alike.*

To these fundamental differences then, which the student already acquainted with the principles of the infinitesimal calculus will have no difficulty in understanding, may be traced the differences which when added backwards from the infinitely small to the finite become the integral that we observe in the living organism.†

As Emerson felt and well expressed it, 'All things are in a scale, and start where we will ascend and descend; and what we are pleased to call results are in truth beginnings.'

* It is worth noting that Huygens at first could see no meaning in Leibnitz's second differentials, still less no doubt in his *n*th.

† The recent abandonment of the infinitesimal, in favour of regarding points, instants, and events as the real entities, does not detract from but add to the notion of a monad.

This point will receive further elaboration in the sequel. Monads group themselves into various grades or scales of being, and the living organism is but a host of millions or billions, trillions, nay, an apparent infinite number, under the control of one that forms the self-conscious unit which in unison with a particular group controls and animates it, as it were; this unit being the soul of the individual.

In unceasing realization of the idea or monad which has taken the lead, the full awakening of consciousness bodies forth the shapes of things unknown and, like a baseless fabric, this vision in co-ordinated forms, gives to the matured Idea a beauty and a truth and value which are its own.

CHAPTER XVIII

THE MONADOLOGY

§ 1

'ALL simple substances,' says Leibnitz in the *Monadology*, sec. 18, 'All simple substances or created Monads might be called *Entelechies*, for they have in them a certain perfection (ἔχουσί τι ἐντελές); they have a certain self-sufficiency or absolute self-power, autarchy (αὐτάρκεια) which makes them the sources of their internal activities and, so to speak, incorporal automata' (Theod. 87). But I maintain only for a selective harmony.

The monad is nothing but a *simple* substance, which enters into compounds. By simple is meant 'without parts' (Theod. 10, *Monadology*, sec. 1).

'The world is entirely in each of its parts, but more distinctly in some than in others' (*Lettre à la Princesse Sophie* (1696), E. vii, 544).

Latta has given us an admirable exposition of the monadology* and Leibnitz's philosophy.

'According to Leibnitz,' he tells us, 'all perceptions are more or less perfect representations of objects, but they vary infinitely in their degrees of distinctness or confusedness. Confusedness is simply a low degree of distinctness, the more perfect any perception or representation is, the more distinct is it, while the less perfect it is, the more is it confused. Thus the differences among the *monads* consist entirely in the various degrees of perfection or distinctness with which they perceive or represent the Universe. But as each *Monad* actually represents the whole universe, however confusedly or imperfectly, and as each is essentially a force or living principle, proceeding, by its own spontaneous activity, from one perception to another, the distinct and the confused are not essentially separate from one another, but it is possible for the confused perception to unfold into distinctness. Each *Monad* contains the whole more or less confusedly within itself, and by its appetition may rise to a more perfect state. Each *Monad* contains as it were enfolded within itself all that it is to be. It is "big with the future". It is like an exceedingly condensed algebraical statement which can be indefinitely expanded, somewhat like the symbol π in the problem of determining the relation between the length of the diameter and circumference of a circle, with this very important difference, that the *Monad* "reads *itself* off". An omniscient Being could see the reality and history of the whole universe within the lowest *Monad*.' The monad is the infinite condensed to a point, but only a sub-group of the infinite with which its ideas have a one-one relation.
(See p. 208.)

* *Leibnitz and the Monadology*, by Robert Latta (Clarendon Press), 1898.

The overwhelming difficulty, as it at first appears, between material and psychic phenomena is thus surmounted, when matter is viewed as a mere perception idea, still more so as a monad. This is the higher synthesis of the two antinomies. That matter should ever act upon mind; that the mere movement of atoms could produce a sensation of colour or sound, compels even the thoroughgoing materialist to assume that the two classes of phenomena run parallel concomitantly as two facets or aspects of reality. Yet this compromise, for it is little else, does not overcome or even touch the difficulty of the insurmountable dualism. Only in the conception of substance which Idealism affords is this dualism converted into a Monism which transcends the apparent contradiction in the material of psychic properties of substance. The *inconvertibility* of the mere mechanism into consciousness, coupled with the incontrovertible fact of the *unity of that consciousness* when so produced, is the point which rends materialism asunder. Added to this is the *freedom of will* of that consciousness, by which the activity of the conscious unit is guaranteed, even as much as the possibility of knowledge is rendered possible by the consciousness of the Ego, as the uniting principle, in unison with its group.*

This apparent inconvertibility of matter into consciousness, coupled with freedom of the Ego, its unity, and power of apperception, upon which as the subject of thought all objects of thought are rendered possible, will occupy our special attention hereafter. It is sufficient for our purpose here to bear in mind that the dualism is thus surmounted; and that, in whatever aspects the incompatibility becomes apparent it is because of the error of attributing to the idea, and particularly the idea of the constituent particles of a dynamical system of Nature, something which it is not, and not recognizing that the idea contains within itself the rest of creation by a one-one relationship between internal or subjective ideas and external objects likewise ideas or monads. Each idea is a separate creation, and yet something which is related to everything else. There are degrees as well as infinitesimal orders of diffuseness or clearness of perception contained in each idea, the monad of Leibnitz representing various grades of consciousness and self-consciousness.

But it is an incoherency in the Leibnitzian thought that the very ideas which form the objects of perception should be denied that consciousness, in varying degrees, which the Egos and other objects of perception of the material world, and the atoms which

* Mind may be regarded as negative matter, see p. 124. For every idea there is a corresponding monad or atom, and vice versa.

make up its structure and composition, have ascribed to them. The Platonic Ideas which I regard as monads are as much so as atoms, themselves their negative correlatives.

§ 2

It is here that the extension of the principles of the monadology to the theory of ideas becomes possible and, I think, necessary; that the theory of the self-existent idea becomes fused with the ever-changing notion of being whilst retaining its separate unity of existence as a separate being.

All other theories are insufficient and self-contradictory owing to the imperfect conception of the true nature and meaning of the idea.

The great obstacle in the way of the Platonic Theory is thus overcome in the higher synthesis of the idea with the monad, which is but an idea of a particular kind. Any other aspect of the idea involves a contradiction. And if to be perceived is the very essence of existence, as Berkeley has proved, it is not more so than to perceive is the essence of independent existence.

The atom is an independent idea, endowed by Leibnitz with the properties of perceiving as well as of being perceived. The vitally conscious beings around us are only atoms in which the faculty of perception is more deeply aroused. But just as we may and do postulate consciousness in the beings around us, which seem to us conscious and self-conscious, and attribute by the same continuity of thought consciousness in varying degrees to the atoms, mere ideas, which underlie the phenomena of nature: so in fact may we, and as our endeavour is to prove, must we from the very nature of reality, ascribe those same qualities in strict continuity of thought, not only to the selves of our experience, and the constituent ideas which compose our idea of the universe of our experience, but to each and every idea which is an object of thought or perception and which accordingly, as truth, has an independent and permanent existence as an idea when so united. It is only the perspective of the idea that forms intensity of consciousness. The notion of infinitesimals in various orders is a necessary part of the conception of the idea, the degree or extent to which self-perception may be applied to it being a matter of the relationship between it and other ideas. For this reason the idea had better be described as the logos of ideology, as the highest synthesis and true development of the monadology. The monad responds to the Ideas, as well as to the so-called material monads. Thus a

synthesis is effected from the monadology and the Platonic theory of Ideas.*

In this modern Platonism as herein propounded we find a distinct leaning, without bias I hope, towards the fundamental tenet of the Christian faith. The doctrine of the Incarnation is shown to be no absurdity; for amongst the infinite host of monads, that which is itself infinite in comprehension, as the greatest of the monads, would sooner or later, and perhaps occasionally appear with matter. The chance, however, that it should appear as man is infinitely small; but extended throughout an infinite time, matter and infinite intelligence might become grouped in one.

The pre-established harmony assumed by Leibnitz has been shown by James Ward and others to be unnecessary in a theory of monads. According to Leibnitz, if there were no pre-established harmony then 'The phenomenon of different minds would not harmonize, and there would be as many systems as there are substances; i.e. it would be entirely a matter of chance if they were sometimes in harmony.'

This is in fact just what Einstein's theory implies, as already pointed out by Robb. But even if monads were distributed according to the laws of chance in time, the overwhelming number would range round an average time as in the case of the velocities of molecules in the Kinetic Theory of Gases, obeying the statistical or exponential law of probabilities.†

§ 3

Leibnitz remarks that all existing objects are related such that their order is 'either *simultaneous* or *before* or *after* each other'.

Time is the order of non-simultaneous existence. It is thus a general order of change in which the determined kind of change is described or considered.

Duration is the length of time. If the length of time be gradually and uniformly diminished, time is reduced to a moment or instant without magnitude, equivalent to a point in space. But in the case of a body in motion there is a correspondence or a one-one relationship between points and instants.

Space is the order of coexistence or the order of existence for all things that are simultaneous.

Extension is the magnitude of space.

According to Leibnitz, 'Space is to be regarded, not as the

* I do not know how far Benedetto Croce would agree with this thoroughgoing Platonism or Platonic Monadology.

† *Vide* N. R. Campbell, *Phil. Mag.*, May 1926.

mutual exclusiveness of real substance, but as simply *the order* of coexistence pre-supposed in the aggregation or grouping of phenomenal things, while *time* is the order of sequence of phenomena' (Latta, *Leibnitz's Monadology*, p. 101).

In a letter to Clarke (Gerhart, vol. vii, p. 363, also quoted by Latta) he says: 'I hold space to be something *merely relative*, as time is: I hold it to be *an order of coexistence*, as time is *an order of succession*. For space denotes, in terms of possibility, an order of things which exist at the same time, considered as existing together, without inquiry into their particular manner of existing. And when many things are seen together one perceives that order of things among themselves.'

Latta summarizes Leibnitz's theory (loc. cit., p. 104), 'Space and time are orders or systems of connexion between phenomena, the bond being coexistence in the one case, succession in the other.'

'Apart from the phenomena, space and time are mere abstractions.' Kant, however, has shown that they are *a priori* sensuous forms of intuition. 'Thus *pure* space and *pure* time are two removes from reality, for the things which are in space and time are not monads but phenomena.' In this it will be seen that Leibnitz was at fault. Latta adds: 'Yet ultimately phenomena are imperfect realities, unanalysable perceptions. They have a basis in simple substance. Thus there must be something non-spatial and non-temporal of which space and time are the imperfect expressions.' Such ideas show remarkable astuteness, even if they do not altogether correspond with the more developed and maturer views of the subject.*

As Kant well puts it: *Time* is a series, *space* is an *aggregate*.

The following letter from Leibnitz to Clarke (§ 47, fifth letter to Clarke) gives Leibnitz's views of the nature of space, in a manner that reveals his masterly power of clear thought and exposition (the translation is taken from Latta's *Monadology of Leibnitz*, Appendix B).

'I will here show how men come to form to themselves the notion of space. They consider that many things exist at once, and they observe in them a certain order of coexistence, according to which the relation of one thing to another is more or less simple. The order is their situation or distance. When it happens that one of those coexistent things changes its relation to a multitude of others, without their changing their relations among themselves, and that another thing, newly come, acquires the same relation to the others that the former had, we then say it is come into the *place* of the other, and this change we call a *motion* in that body, wherein is the immediate cause of the change.

* But even these apparently indistinct ideas are realities.

'And though several, or even all the coexistent things should change according to certain known rules of direction and velocity, we can always determine the relation of situation which each acquires with reference to every other; and we can even determine the relation which any other (coexistent) would have (to this), or which this would have to any other, if it had not changed, or if it had changed otherwise.

'And supposing or feigning that among those coexistences there is a sufficient number of them, which have undergone no change, then we may say that those which now have to those fixed existents a relation such as that which others formerly had to them, have the same place which these latter had. And that which comprehends all the places is called *space*; which shows that in order to have an idea of place, and consequently of space, it is sufficient to consider these relations, and the rules of their changes, without needing to fancy any reality outside of the things whose situation we consider. And to give a kind of definition: *place* is that which we say is the same for A as for B, when the relation of coexistence between B and C, E, F, G, etc., is in perfect agreement with the relation of coexistence which A formerly had with the same C, E, F, G, etc., provided that in C, E, F, G, etc. there has been no cause of change . . . *Place* is that which is the same in different moments to different existent things, when the relations of coexistence between each and certain other existents, which are supposed to continue *fixed* from moment to moment, agree entirely together. And *fixed existents* are those in which there has been no cause of change in the order of their coexistence with others, or (which is the same thing) in which there has been no motion. In short, space is that which results from places taken together.

And here it is right to consider the difference between place and the relation of situation which is in the body occupying the place.

For the place of A and B is the same; whereas the relation of A to the fixed bodies is not exactly and individually the same as the relation which B (that comes into its place) will have to the same fixed bodies: these relations are only in agreement. For two different objects, as A and B, cannot have the same individual affection; it being impossible that the same individual accident should be in two objects or pass from one object to another. But the mind, not satisfied with mere agreement, looks for an identity, for something which should be really the same, and conceives it as outside the objects; and this is what we here call *place* or *space*. But this can only be an ideal thing, involving a certain order, in which the mind conceives relations to be applied.'

We may therefore regard place or position as a property or characteristic of superposition or coincidence in space, as in the quantum mechanics what the Germans call *Beisammensein* (or being together) (see Clauberg and Dubislav, *Systematisches Wörterbuch der Philosophie*). It includes therefore not only the quantity or magnitude or size or space, but its quality. This magnitude or size is that characteristic of things which is in them only through the immediate simultaneity of coincidence, or through the simultaneous perception by which they are recognized.

The *quality*, on the other hand, is that characteristic of things

which admits of their recognition when one considers them singly, and without that, they would only be given in immediate coincidence.

Things which have the same quantity are equal.

Things that have the same quality are similar.

But as Bradley has shown, in the limit, similarity when properly analysed means identity of quality, else a contradiction would ensue.

The perception of an object in space and time implies a relationship of the object to others of the manifold of the space-time 'event'. But in so far as it is analysable it is itself in time the subject of the qualities ascribed to it, or prior to them as object. And in the inevitable relationship of subject and object it is itself of the nature of an idea.

§ 4

The word 'Idea', when carefully considered, does not admit of any very clear definition, and there are authorities who would prefer not to use the word at all. Its use can be justified only by giving it a precise signification, in the particular sense in which we desire to employ it for our special purpose. Otherwise nothing but vagueness could ensue.

The idea of an object as the substance of thought corresponds to the external reality which is also an idea, the development of which follows the law not of association in our minds, but those of the nature of the idea itself. Yet we perceive not copies of ideas, but ideas themselves.

In every case the definition and reduction of ideas to their simple and simplest elements is an essential procedure. In their ultimate analysis, it is found that identities result. This then brings us to the most modern conception of the Association of Ideas and of Inference, somewhat on the lines of F. H. Bradley's principles, though differing largely from them. 'The same and yet not the same', except in the sense in which 'sameness' is used by him—a perfectly legitimate use of the word when dealing with the identity and essence of things. For the present it is sufficient to state that the method as well as results are entirely independent of Bradley's way of looking at things or his manner of treating them. We must first consider the monad in its extended sense.

Although it is to Leibnitz that we owe the conception of the monad, or unit of consciousness, it must nevertheless be pointed out that his *Monadology* involves some strange contradictions, which his method, if it had been sufficiently developed, would have readily disclosed. His monad is 'pure substance', the *primal essence*, or

ultima thule of things. And yet it had for its substance, not the idea, but a complex of ideas. If Leibnitz had been consistent he would have perceived that the true monad or unit of substance is the idea itself: separate and yet united, as the one with the many in the continuity of subject and object in the eternal nature of things.

As we have seen:

Space is the order of coexistence of monads forming an event.

Time is the order of succession of 'events'.

An *Event* is the *apperception*, or as we should say nowadays, the *awareness* of the order of coexistences at any instant.

Motion is change of coexistence with time.

Force is change of motion.

Space is three-dimensional, time is one-dimensional and unidirectional. Thought as regards objects and events, except in so far as it is confined to space and time, may be considered either as non-dimensional or else of as many dimensions as there are terms involved.

Leibnitz's ideas upon these subjects are not as clear as one might have wished. Consistency and continuity of thought should have led him to regard the ultimate ideas or constituents of thought as monads. But apparently he regarded only certain units as monads, such as men, animals, in general, possibly plants, and certainly the atoms of which they are composed, as units or centres of force, with varying degrees of consciousness, each such monad 'mirroring' the whole universe within itself.

Force, though the characteristic property of the monad, is not an attraction or repulsion between bodies; for the monads are in fact quite self-contained, independent and intact, so that they are not supposed in any way to interact. The attraction or repulsion is only apparent, and 'force' is in reality non-existent, although he emphasizes that monads are points or centres of force! There is harmony between bodies, and the forces between them are apparent and not actual.

This view is quite in accordance with the modern theory of relativity, as Professor Wildon Carr has shown.*

For the purposes of our present investigation we are compelled to go through sheer consistency to extremes, and by perseverance to join issue where extremes meet.

If the monad is composed of ideas, then it cannot be pure substance itself. Whereas if it is the central idea in its own universe of thought, 'mirroring' as it were a greater universe of monads like

* *The Monadology of Leibnitz* (1931).

itself, the units of its own substance of thought ought to be, on his own principles, monads as well. Not merely other beings, and other bodies around him, down to the atoms of matter, but the ideas of which each and all are composed are monads in their own degree; each being subject and object, in a dimly conscious unity in which it mirrors the rest of reality, and with which it is connected by the continuous chain of subject and object *ad infinitum*, a chain in which it occupies the comparatively simple and perhaps humble role of a mere link with the rest of things, but without which the whole would be no more real than the part.

The relationship of subject to object as a logical connexion of ideas everywhere holds in a greater or less degree of juxtaposition; with greater or less possibilities of development, according to the nature of the idea or ideas involved.

§ 5

When we regard the idea of an object as coming under the category of space and time, the relationship of subject and predicate is of such a nature that the physical properties come under consideration as the necessary attributes of the body in virtue of its presence in space and time. Its physical properties are merely the relationships so expressed. But the biological properties, though ultimately physical, involve a factor which ordinary physical phenomena disregard.

The *cyclic* nature of biological processes does not necessarily apply to the purely physical. The ends of the chain of ideas do not return into themselves in time. But in space, positive and negative electricity, the constituents of matter, always tend to unite. Their separation in space only introduces the relationship of distance. By a fundamental conception in physics the bodies or particles come together according to the *Principle of Least Action*, which is explained elsewhere.

The physical relationships expressed by mathematical equations are merely an extension of the relationship of subject and object from the logical and metaphysical aspects. The entities contemplated being merely ideas admitting of development along particular lines, the development of the properties of certain classes of ideas ceases to be merely analogical, and becomes actual reality as monads coexisting in space, that is, having the relationships in accordance with the Space Category relatively to other monads. These are, in fact, the particular objects familiarly designated as the material universe of our perception.

To Leibnitz, Lotze, and Ward the monad is the essence of

reality. But the conception of the idea as the true monad gives a unity and consistency to the whole, which brings us back to the fundamental notions involved in the Platonic theory of ideas, towards which F. H. Bradley has pointed the way.

A definition of the word 'idea', that is of an idea of an idea of an idea *ad infinitum*, is thus no very easy matter. Yet our discussion and investigations rest upon it, and of all definitions it is perhaps the one in which a definition in a circle can only with the greatest difficulty be avoided. It would, perhaps, require a review of the whole history of philosophy from Thales to Bradley. But the physical aspect of the question, the analysis by our characteristic methods and the application to scientific problems, with the results thereby evolved, renders it unnecessary to discuss the various meanings in which the word has been used; and it is sufficient to confine the word to the meaning in which we intend to employ it.

An 'idea' may be defined as 'the unit of the substance of thought'. Or it may also be regarded as 'the unit of perception' whether of 'thought', of 'feeling', or of 'action' or 'will'.

Now each one of these terms might form the subject of a separate treatise in the sole endeavour to define it. And the perennial embarrassment of philosophers can be traced to the meaning they attach to these words.

Following Berkeley, we may say that 'an idea is a perception'.

A proposition which, of course, is an idea is nevertheless a juxtaposition of two ideas that apparently overlap, but which so far as they have something in common are identical in this respect, however much they might differ in other respects.

It is here that Boole's method comes in useful, and that its development becomes strikingly apparent.

In common parlance an 'idea' is regarded as a 'mental picture'. But this is a composite idea agreeing more with Hume's limitation of an idea to a more or less mental reconstruction of perceptions or impressions. The perception or the percipient process described as an 'impression', even in its simplest form, is to be regarded as an idea, with all possible combinations thereof. The reconstruction or 'reproduction with a more or less adequate image of an object not actually present to the senses' after G. F. Stout and J. M. Baldwin (*Dictionary of Philosophy and Psychology*) limits the meaning too much, for the more general signification of the term.

I shall use the word in Berkeley's sense, so as to include any

and every mental impression irrespectively of its nature as to complexity or simplicity and in whatever manner it may be produced, internally or externally, whether by the objects of sense or by the mind itself.

The combination of two such ideas and their relationship is an idea of a general type expressible as a proposition or judgement the predicate of which Plato regarded as a 'universal'. He uses the word for the 'eternally existing reality, the archetypes of which the objects of sense are more or less copies'. In the theory here put forward they are not images, but the things themselves with which a monad is in tune. This definition implies so much that it cannot be taken as a definition to start with, even if it should afterwards, as we shall show, turn out to be true; for the use of the word 'reality' is here an objection to our purpose.*

§ 6

In the middle of the last century the scientific mind was shocked at the Hegelian attempt to identify the thing with the idea. Material atoms were regarded as centres of force possessing inertia. Yet to-day force is regarded merely as the result of relative motion; and matter, as Prospero foresaw, 'melts into air, into thin air, leaving not a rack behind'.

The Faraday lines of force between electron and proton, the lines of magnetic force between magnet and magnet, the lines of gravitational force between sun and planet, only reveal the properties of space more clearly than would our vision or our touch.

Every atom, every idea contains within itself the secret of the apparent mystery of its being, which in suitable circumstances enables it to unfold the story of its nature. Ideas, no doubt, are of all sorts or degrees and manner of complexity. The highest we know of are the human monads, our own Ego, which we perceive directly, and the Egos of others which we infer indirectly, whilst the lowest in the material world is the photon, also inferred though somewhat more indirectly: yet there cannot be a limit; for limit there is none, and the scales may be projected in thought in both directions; from the infinitely great to the infinitely small, in one continuous and harmonious whole 'whose body Nature is and God the soul'. But in truth the idea forms at once the centre and the circumference, the individual and the all, the unity and the

* An *Idea* is a '*perception*' a part or the whole of which one is '*aware*'. A *conception* is a perception of a universal or *abstract idea* as distinct from a *particular* or *individual*.

totality, the one and the many, the agreement and the difference, the continuity and the discontinuity, the subject and object in one; though chance and circumstance may have served to draw out certain types; but the inviolable relationships of truth have moulded and guided the evolution from that chaos we call nebular matter to the self-conscious and deeply conscious creature that perceives this comprehensive unity within it, as a monad, as though an 'image' of the whole; the one, the true, the good, the beautiful, the sublime; and yet not an image, but a percipient part of the great reality.

Bishop Berkeley laid the foundation of the modern theory of the subjectiveness of space. His argument seemed to him to apply equally to time, which he regarded as nothing but the succession of ideas. But a profound difference does nevertheless exist between these two views, even from the idealistic standpoint. 'Our conception of time', says Henry Sturt, the author of *Personal Idealism*, 'is based upon our experience of change, combined with memory and anticipation. Now change is an experience which we feel directly in our personal consciousness, consciousness is *not spatial* but it is *mutable*.' 'This direct experience', he adds, 'is a guarantee of the realness of change, and justifies us in attributing it in some degree to ultimate reality.' This in fact is just what M. Bergson has endeavoured to do in his *Creative Evolution*, in *Time and Free-Will*, and in *Matter and Memory*. But the reality of time is no guarantee of its universality, and its subjectiveness in consciousness still remains, so that one man's time is not necessarily another's. This Bergson overlooks, and it is where more recent ideas gain a march upon the older conceptions. A *universal now* is not a necessity, so far as the individual units or monads of consciousness are concerned, even if it should prove to be a metaphysical reality.

So far as our experience goes, consciousness is the real thing. But that it is *mutable*, no more testifies to the reality of time apart from consciousness, than the sense of place, or the sense of motion, could be said to testify to the reality of space. The 'flux of consciousness' is what is real in consciousness as a 'psychic state'. We cannot say that it flows at a perfectly uniform rate without combining it with the measure of movements in space, along equal distances, as uniform velocity. But even this assumes equalities which may be inequalities. Therefore equality is something which it is entirely impossible to ascertain. The unit of length may change, and according to the theory of relativity does change, and the unit of time with it.

The relationship of subject and object as regards 'material' monads whether organic or inorganic, is in a 'space-time continuum' which fulfils the Einstein *invariant* interval between different coordinates of reference.

Now let us observe that if the subject-object relationship alters the duration or sense of time in a particular monad, and such a change under the directive power of will on thought is possible; then for the sake of argument, this may be for the present granted. Such a change in time, according to the theory of relativity, would be accompanied by a corresponding change of length, so that two monads would appear to move towards or away from each other as though under the influence of a force.

Hence the physical explanation of the action of mind upon matter. For if the monad has the power of altering its *sense of duration* or rate of succession of its ideas, that is its time, according to its own subjective standards, such a change would become manifest as a force on the adjoining monads, namely those composing its material body or organism. Will-power over the material organism thus becomes the power of regulating the subjective rate of succession of ideas. The action is therefore not on the matter at all, but upon the mind itself.

Without going into the question in detail for the present, it is clear that not only from a physical but from a physiological and psychological point of view this conclusion, though based upon the assumption that the physical duration is subject to the will, is one of supreme importance from a philosophic standpoint.

If the time of a monad is very greatly altered the force between it and other monads may be considerable during the change. It would in fact be possible for a medium as it falls into a trance to manifest some such effect as that of telekinesis. Further, if the transition into this state of trance or hypnotic state is accompanied, as Cassali of Turin has found, by electromagnetic radiation which can be detected in the ordinary manner by coherers and electric valves, then telepathy becomes readily intelligible. Further, the experiments of Sauerfach and Schuman of Vienna seem to show that varying electromagnetic radiations accompany emotions.

Apart from the results of psychical research which are on the whole, as we have seen, not regarded as strictly of a scientific nature and rigour, there is the consideration of time in relation to free-will on the lines indicated by Bergson, though not necessarily involving the universality of time postulated by him. He supposes that creation by will-power affects matter by putting it into motion; whereas consistency would have led him to infer that if the will

could create, it could equally affect the rate of succession of ideas, i.e. time, as a reality in the conscious effort, rather than act directly upon matter.

If a man happened to have the power of altering his time in the 'space-time continuum' he would, of course, be able to put himself into motion. That is, the action of his will would be on the idea, as related to time and space, rather than on matter. The difficulty of explaining the apparent action of mind upon matter would thus be overcome, for the action of mind would in fact be, not on matter, but on the idea of time, which is merely the rate of change of events, or rate of succession of ideas.

There is at first, however, a difference between physical and psychic time. The 'physical time' is the rate of succession of events in the 'space-time continuum'.

'Psychic time' is the rate of succession of ideas in the individual mind.

Hence the psychologically indissoluble nature of 'cognition' and 'conation' would be explained, if we could but assure ourselves that 'physical' time is in fact 'psychic' time.

Cognition and conation are, as we say, psychologically inseparable. But we know as a fact of everyday experience that the effort of will or conation does, at any rate as regards the body of the individual, apparently act upon matter. And we have seen that, according to the theory of relativity, conation becomes a mere act of cognition, if physical and psychic time are one. Conversely, the psychological fact that cognition and conation are inseparable may serve to establish an invariable connexion between physical and psychic time. We know that the 'behaviour' of the psychic unit in its normal states is in accordance with the laws of the material universe.

And if we were to admit some of the well-known 'facts' of levitation or telekinesis the case would be proved. Unfortunately, we cannot enter here into the validity of the evidence for this very remarkable phenomenon.

A simpler and more familiar fact, however, is when a normal person puts himself into motion by the effort of his own will, and thereby alters not only his space relations but his time relations to other bodies. Here, of course, there is actual contact, though the action is of mind upon matter. If will-power is accompanied by an electro-magnetic radiation, as Cassali claims to have proved, then the action of will-power through space becomes intelligible. The amount of energy required to produce the alleged effects of levitation would be enormous, which, if not impossible, would at least

render the effects extremely improbable, even in the light of the above considerations. That 'faith can move mountains' is thus seen not to be an impossibility, but the will-power and energy required would be entirely super-human, unless it should be found that like the radioactive atom the human monad has a similarly large store of energy within it—a possibility which cannot be denied—and that in the hypnotized state it admits of being drawn upon. This again cannot at present be affirmed or denied, from the strictly scientific standpoint. But its possibility must be admitted.

One thing is certain in this connexion and that is, that if an electromagnetic radiation is emitted by a hypnotized subject as it enters the state of trance, then its physical time is altered, whether it can be identified with its psychic time or not.

It is an established fact that by an effort of will not only the temperature of one part of the body can be made higher than that of another, but impressions made. [The most remarkable cases, of course, are where actual sores are produced, 'stigmata'.] Now if the temperature of different parts of the body can thus be altered by the will, and it is quite common in highly strung or hysterical persons for the two hands or two feet to be at very marked differences of temperature, such persons would thus radiate slightly more from one side of the body than from the other and therefore more in one direction than in another. They would therefore experience *a force however small* in the opposite direction and accordingly again, to however small an extent a *change of time—actual physical time*. Such people are accordingly somewhat out of tune, so to speak, with the average 'space-time continuum' of normal human beings around them.

The establishment of this fact independently leads to the relationship between cognition and conation, one of the fundamental facts of psychology.

The change in time produced may be quite infinitesimal but it would be none the less real. Vice versa, the action of mind upon matter would involve radiation in opposite directions. Dualism is avoided. The mind itself alters its most intimate connexion with reality by affecting the rate of succession of its ideas, not in mere imagination, but in actual reality in the 'space-time continuum'. For in its close grip with the 'idea' of time and endeavour to grapple with it, it loses some of its energy.

This 'physical time' is purely relative, since it depends upon the relative motion and position of bodies, and therefore belongs to the 'space-time continuum'. Absolute time is a metaphysical

notion and has nothing whatever to do with the time of the physicist, which is measurable relatively to the physical units of his choice, which, as we have seen, are variable and relative. The absolute *Now* cannot be altered by any effort of consciousness, relative motion, or change of place. It is an absolute *now*, here and everywhere, with no possibility of duplicity except by altering the meaning of the word. It has no physical signification, other than as indicating a universal *now* with a *before* and *after*. Such absolute time is no more measurable than absolute space. It can neither help to define our ideas nor in any way to fix, clear or refine them. It is in fact Platonic time, like Absolute or abstract truth itself.

The physical, psychological, and philosophical aspects of time and space are accordingly different; there is also, as we shall see, the mathematical aspect, but none of these when rightly understood really contradict each other, since they are not necessarily incompatible, and in fact help to throw light upon the different notions they represent. The theory of relativity as developed by Einstein has regard to relative motion referred to different systems of co-ordinate axes and, as we have seen, the units of space and time are accordingly altered. This change of dimension is as significant as that of time.

§ 7

There is another aspect of relativity which this theory does not take into account, and that is that *not only is all motion relative, but magnitude is likewise so*.

Just as there is no absolute space and no absolute time, *physically speaking*, so there is no absolute unit of dimension. In fact in the same system of co-ordinates there may be, as we have seen, a continual change in the magnitude of the units, relatively to some absolute value, if there be such, though it be quite unknown to us, that is, a continuous increase or a continuous diminution, or even *an irregular or abrupt variation* in these, that cannot be detected directly. Everything may be, for anything we know to the contrary, continually expanding, or continually contracting, or getting continually faster, or slower, or varying haphazardly, or according to some definite law, without our being able to detect it, so long as the whole Universe *as we know it* undergoes the change in the same proportion. It apparently makes no difference to us whether this is so or not, so long as everything changes in the same proportion. Moreover, our ideas of space and time are of such a nature that apparently there is an infinity both ways, from the infinitely large to the infinitely small. And just as it appeared in pre-Einstein

days that a body moving in a particular direction might go on moving in that direction to infinity for all time, so is it possible that it might go on expanding, or diminishing, so long as everything else did likewise, for all time into the infinitely large, or the infinitely small, relatively to our own scale of being, without any change being apparently produced. Time and space would still be the same whatever alteration they might undergo relatively to other universes, or systems of units of which we know nothing at present.

I propose to call this 'Fluxional Relativity' or Differential Relativity to distinguish it from the 'special' and 'general' theories of relativity of Lorenz-FitzGerald on the one hand, and of Einstein on the other.

CHAPTER XIX

THE NATURE OF TIME—THE ACTION OF MIND UPON MATTER

THE difficulty we have to contend with is that time being intimately connected with change, this liability to change, which is its very essence, seems to render objects unreal, as contrasted with the permanent reality of the thing-in-itself of the materialist, and the universal ideas of the Platonist idealist, or even many if not most modern idealists. Henry Sturt remarks, 'The pantheistic One-and-All of the Neo-Platonist, Plotinus, is plainly incompatible with the reality of Time. In all pantheistic systems Time belongs to mundane existence and Eternity to the transcendental Reality.' M. Bergson could not possibly accept this view of time and reality. But the modifications we propose in the higher Hegelian synthesis of this problem render the creative conception of time feasible according to the more modern ideas of relativity; whilst not disputing the more ancient Neo-Platonic notion of reality which in truth they ultimately tend to confirm, the two are found to be in accord when the differences between them are properly understood.

M. Bergson makes change the reality, whereas it is the changing entity that is the reality.

If, for instance, the sense of duration were indefinitely diminished, and it is quite conceivable that it should be under certain circumstances, just as when the motion of a train is slowed down, until it is at rest relatively to surrounding objects, there would be in such a case a consciousness of rest, in one case in time, and in the other in space; duration being merely the rate of succession of ideas. Hence if one idea remained fixed in the mind, by the gradual dimness of the sense of succession as in certain psychical states of Buddhistic ecstasy, the sense of permanence might conceivably be reached, a sort of hypnotic state, or waking dream, often described not only in Oriental literature but in European *belles lettres*; nowhere indeed more admirably than by Goethe in the 'Confessions of a Beautiful Soul', in *Wilhelm Meister*; or by Tennyson in *The Holy Grail*, and in *The Ancient Sage*:

> In moments when he feels he cannot die,
> And knows himself no vision to himself.

The flow or flux of ideas is in fact a way of thinking, and a person who was always in a moving train would find it inconceivable to imagine a state of things in which there was rest, whilst one who never moved would think motion inconceivable.

Consciousness therefore does not necessarily mean existence in time, but existence in time does mean activity and change, so that cognition and conation become inseparably united once more in the manner the normal psychology requires.

Activity implies the power to change, or conation, whether in time alone, or in space and time, which is *motion*. In normal psychological states there is no cognition without conation. But if it be conceded that in abnormal states cognition with an infinitesimally small amount of conation is possible, then time becomes no whit more essential to consciousness than motion. A most important point. Except in the infinite extremes the two are inseparably connected, so that as cognition affects conation in thought, i.e. will; so, vice versa, will can affect thought and with it the sense of duration or the sense of time.

In order to avoid many familiar misconceptions and fallacies it should be emphasized again that the word time may be used above in three different senses. Much futile discussion may be avoided by recognizing the differences between them. (1) Absolute time; (2) Physical time, which is relative but partly subjective; (3) Psychological time, purely subjective.

(1) Absolute time, though implying a definite *Now* common throughout the Universe, can no more be determined physically than absolute space. The reality of either cannot be denied and need not be asserted, for in physical measurement they do not enter into experimental considerations. From a metaphysical standpoint the idea of absolute time is of importance. It is not necessarily inconsistent with Idealism, for even if time be subjective it may be common to all minds, and, like truth itself, be a universal reality.

(2) Physical time, however, depends upon simultaneity and the measurement of equal intervals, both of which are affected by the motion of bodies relatively to each other. Let us suppose three bodies A, B, C, in a straight line, AB being equal to BC. Suppose that a signal be given at B, at let us say midday, so that A and C might set their clocks accordingly.

If A is moving towards B, and C away from it, it is quite clear that the signal will reach A before it reaches C, as the message whether of sound or of light will take time to travel and will have to overtake C, while A comes forward to meet it. In any case, however, both A and C will be after B, and the instants will not only not be simultaneous, but succeed each other in the order, B, A, C, and as there is nothing else to go upon, the clocks at A

will be slower than those at *B*, but tend to gain up to them, while those at *C* will also be slower than *A*, and therefore *B*, and get slower and slower indefinitely. The clocks at *A* will go faster than those at *B*, and these faster than the receding ones at *C*.

Time therefore thus depends upon relative motion. Events do not take place at the same absolute time, but depend upon the place, while the measure of time depends upon the relative motion of bodies. The unit of length changes, and that of time with it.

(3) Psychological time, as we have seen, is purely subjective. This psychological time is what Locke called duration. He made it clear that it is not a measure of time, but the rate of succession of ideas which may vary more rapidly, for instance, under the influence of calenture than in normal moments; and indeed it may be slowed down in moments of distraction, so that an hour may appear as a few minutes, or to the Buddhist as eternity; while the evidence of persons saved from drowning and similar cases shows that a few moments may appear as a lifetime.

The 'specious present' or purely 'psychological present', as Bergson calls it, gives no guarantee of real duration, nor indeed of uniform duration of any kind. Time as a measurable quantity cannot be reckoned without space. The two must be considered together as in the 'space-time continuum' of the physicist. But in so doing it still remains 'subjective'. This corresponds to Bergson's 'spatialized time'.

He distinguishes between the 'specious present' which, as he supposes, gives real duration or real time, though, as a matter of fact, it cannot be measured and is in truth subjectively apprehended by 'intuition' as a continuous process; he distinguishes between this and the concept of 'elapsed' time formed by the union of the concept of succession derived from actual experience of duration and the concept of distinctness given by the experience of space. This he calls 'spatialized' time. The latter is presumably physical time. But Bergson does not enter into the physics of the question, else his 'clean cut' section of reality as the concept of time whether 'specious' or 'spatialized' would render his idea of time absolute. The Bergsonian idea of time, however, need not be a universal time, though he regards it as such, since each monad may still have its own time, which corresponds to a section of *its own* reality.

The monads need not beat time together as by a 'pre-established harmony', and yet be 'specialized' and 'individualized' at once. The psychical time is mere cognition, whereas the element of conation gives it a space relationship, such that the inviolable connexion between cognition and conation renders the thought process in

time one with action, so that time and will are related as intimately with each other as will and space; and it is a matter of indifference whether we regard the mind as acting upon matter and causing it to move in space, thereby altering the time, or suppose it to regulate the time so as to produce the effect of motion in the 'space-time continuum'. Conation then becomes the objective manifestation of the effort of the mind to alter the rate of succession of ideas, not through the memory or the imagination, in the 'specious present', but in virtue of its 'spatialized' time connexion in the 'space-time continuum' which is manifested by its apparent action (indirectly) upon matter.

'Pressure', 'Force', 'Impenetrability', 'Resistance', these are terms that stand for the consciousness of the action upon matter, or the reaction offered by it. They are the physical as well as psychical concomitants of the mental control of ideas of a particular kind; so that, if an idea is retarded or advanced in its succession to others, the cells in the cortex are acted upon physically, and an 'effect' produced as a *conative* action,which is transmitted through the appropriate channels, resulting in motion of the body and external objects in contact with it.

Bergson's conception of time, though at variance with the theory of relativity, yet admits, by a process of Hegelian synthesis, of modification to a higher interpretation of the meaning of time and cognition, quite in accordance with relativity. We thus arrive at an explanation of the action of mind upon matter, without recourse to dualism, whilst establishing the psycho-physical phenomenon of the inseparability of cognition and conation, or thought and action.

There is a third element that modern psychology regards as likewise inseparable from both of these, and that is 'affection' or *feeling*. Cognition, conation, and affection, i.e. thought, action, and feeling, are one and indivisible, constituting the trinity of consciousness, as the very essence of mind.

This unity attests the presence of the perceiving Ego, to which the so-called reality of each passing event is as a fleeting shadow, or the passing panorama of the train in its flight through space. But the time through which it is realized is itself the expression of the self that perceives it. No flow or rhythm of ideas gives the full content of these perceptions, except when the monad responds to them by virtue of its inner nature in being so 'attuned'. Within itself this feeling, self-perceiving, thinking, active unit, or conscious idea; the Ego, attests its existence by a knowledge of its own self as a self-conscious reality, responding to ideas fitted to its inmost being, performing the function of a self-assertive unity, as a source at once of energy and activity, and a persistent and independent being.

CHAPTER XX

NON-PRE-ESTABLISHED HARMONY

§ 1

BEFORE going further into this matter it will be necessary to consider more fully the ideas or intuitions of space and time from their physical aspect, and thus return to the psychological ones for the elucidation of the views put forward.

We must bear in mind once more that these words are continually used in various senses. Metaphysical time as absolute time is not the measurable time of physics. And physical simultaneity is not the same thing as the Universal *Now*, the events which occur here may not become events at other parts of the Universe for thousands or millions of years to come. For instance, the light emitted at the present moment by Sirius will, as we know, not reach us at the Earth for some forty years to come, when it will appear as a reality realized here. Yet what is taking place there at this moment is in the Universal *Now*, although it may take forty years for the message of the event to reach us.

If the word 'Now' is used to mean the instant of the realization of an event, there would be a difference of forty years between the two 'nows'. Thus physically the 'now' on our Earth will not be 'now' at Sirius for this number of years, and to the inhabitants of some of Sirius' planets, if there be such, and there probably are, the Earth would be passing through the political crisis caused by the introduction of Mr. Gladstone's first Home Rule Bill, and everything would be forty years behind the times. Mr. Parnell and *The Times* would just seem to be coming to blows, as though the times were out of joint; accursed spite if only an increased speed of light could put it right! What must be borne in mind is that the awareness of an event at different parts of the Universe represents the physical or spatialized time, but that it is not to be identified with the absolute time which, though not measurable, nevertheless at each instant corresponds to a universal *now*. Yet this statement is not strictly accurate, for the time at Sirius is not merely a function of the distance between it and the Earth but also of the relative velocity with which the two bodies are moving. Thus

$$t' = \left(t - \frac{v.x}{c^2}\right)\beta,$$

where t is the time here, t' that at Sirius, and x the distance,

while $$1/\beta = \sqrt{\left(1 - \frac{v^2}{c^2}\right)},$$

known as the FitzGerald contraction, the basis of the theory of relativity.

It is an important problem, whether mind can act upon 'spatialized' time in the manner that has been considered. For if time be purely subjective, there is no reason why it should not do so. It is clearly entirely a question of experience.*

If matter be merely energy, as the quantum theory requires, a body that radiates loses matter. Hence if the radiation be unidirectional, which can be effected by mechanical means as in wireless telegraphy and telephony, or indeed when a beam of light is concentrated by means of a concave mirror, the radiating body will experience a force or tendency to move in the opposite direction.

Now if, as we have seen, a hypnotized subject can radiate electro-magnetic waves—and it matters not in the very slightest for our present purpose whether the radiation is really due to thought transference, or merely electro-magnetic disturbances set up by the variations in the electric currents of the nervous system of the subject, arising from the change in its mental state—such electro-magnetic waves, which can be detected by the ordinary means employed in wireless operations, might conceivably be directional, that is the rays emitted through the eyes, for instance, might be less absorbed than those emitted by the back of the head. In such circumstances then the medium should experience a reaction in the opposite direction, small though the effect might be. However minute this effect might be, his time would, from the physicist's point of view, be altered, and he could not be said to think in precisely the same time as before. His 'space-time' line is altered. So likewise another such medium sensitive enough to these radiations—and there is little doubt that some hypnotized subjects are sensitive enough to perceive these—will experience a force in the opposite direction. His time will likewise be altered. If the time be increased in one case it will be diminished in the other, or vice versa, so that two such persons could be made to have the same time! They might therefore be in a position to think alike as if they occupied the same place, and differently from those around them.

Here the physical time is actually altered, and like the psychical

* A suggestion, to this effect has been made by A. A. Robb in his *Absolute Relations of Time and Space*, p. 80 (Camb. Univ. Press, 1921).

time becomes subjective. The abnormal relationship puts the hypnotized individuals in harmony with each other and out of joint with the rest of things. As we have seen, the more familiar phenomenon of a person putting himself into motion by the sheer effort of his own will enables him to change both his space and time relatively to other bodies. The motion is caused by the effort of will, whereas it would equally follow if the effort of will merely acted upon the train of thought in altering not only the sense of duration but, as we have seen, which is quite possible, the time itself in its true physical sense. As we have seen, the psychic or subjective aspect of time is closely related to cognition and conation as inseparably united states. But it now appears that the theory of relativity does in fact enable us in the light of still more recent psycho-physical investigations to identify the time of cognition with that of the physical state.

This may justly be claimed to be a triumph for these abstruse meta-psychical and psycho-physical researches, from the combination of mathematical, physical, and biological studies.

It is indeed difficult to see how if we could ever dispense with the *here* we should not be able to do likewise with the *now*.

That time should be the reality and mind merely something in it, seems to be the very antithesis of the truth. Arguments used to prove one would apply equally to the other. What is true of space in this respect is true also of time without any reservations whatever. If we cannot divest ourselves of the one, it is not easy to see how we can deprive ourselves of the other and vice versa. The idea of time as a clear clean-cut section of reality, emphasized by M. Bergson cannot be conceded on the theory of relativity.

Furthermore, it appears from the considerations set forward above, according to our theory of time and space these are both subjective as conditions of things through which alone they become realized in the monad, mere *forms of sensuous intuition*, as Kant regarded them.

This is clearly not in accordance with Bergson's theory, any more than that the latter accords with the principle of relativity.

Now though time be supposed to be the reality, it is in fact consciousness that is actually so, and time but the peculiar form or mode of the transformations of ideas as they seem to pass within or occur in conjunction with the idea of the self or monad.

Time cannot find expression in thought without the individual's own primary sense or cognition of the changing ideas of succession; and instead of the individual consciousness being a function of time, it is found that time is a function of it.

§ 2

To reiterate what has been said for the sake of clearness, physical time is measured by clocks, and is essentially objective to start with, whereas duration in Locke's sense is purely a subjective effect. In either case there is a common *now*, that is an instant implying a continuous change from a *before* to an *after*, always in the same order of sequence. In the case of physical time the rate of change is measured *locally* by physical phenomena such as the rotation of the Earth on its axis, a period that for practical purposes may be considered to be constant. 'Duration', however, is the rate of succession of ideas in the mind itself and may appear faster or slower according to the disposition of the individual conscious of it. As already stated, a familiar experience is that of lying awake at night, when the hours may seem to be interminably long; the ideas following each other rapidly. On the other hand, it may happen that under similar circumstances in the day time they would appear to be imperceptibly short, or vice versa; the ideas in the latter case following each other slowly according to the particular state of the nervous system in each case.

It has been shown that according to the theory of relativity the 'now' in different places is different, and only under particular circumstances ever identical. But, as we have seen, it is possible also in the light of considerations so far adduced, to conceive that even physical time and the physical 'now' of the conscious subject can at any particular place be different from the physical 'now' of its locality under ordinary circumstances.

Thus it is conceivable that a mind so endowed might project itself into the future and acquire a knowledge of events to come by actual premonition. And, as we have shown, there is evidence on purely physical grounds both for the credibility and the existence of this phenomenon. There have been no doubt many cases of fraud, and the establishment of such facts is extremely difficult, but as in the case of science, the history of alchemy does not render the sciences of physics and chemistry on that account incredible. Scientific incredulity so far has not been due to unscientific prejudice, but to the inherent improbability of the phenomena in the absence of any scientific explanation of such effects. When, however, they can be shown to fit in with scientific ideas, their *a priori* probability is thereby very greatly increased.

Such pre-established harmony as was assumed by Leibnitz has been shown by Lotze and others to be quite unnecessary in a theory of monads. Leibnitz held that if there were no pre-

established harmony 'the phenomena of different minds would not harmonize and there would be as many systems as substances, or rather it would be entirely a matter of chance if they were sometimes in harmony'. This in fact, as we have seen, is just what Einstein's theory virtually implies, as Robb has already emphasized. Yet even if they should have been so distributed perfectly at random, then according to the laws of chance the overwhelming number would range round an average as in the case of the velocities of molecules in the kinetic theory of gases, obeying the exponential law of probabilities.

Now if the monads do act upon each other, although Leibnitz supposed that with metaphysical strictness they could not do so, then those that are approximately harmonious will tend to become more and more so, while those that differed largely would be driven more apart into some other universe of discord and strife. Thus a process of 'natural selection', as it were, would have given rise to a universe or universes of approximately simultaneous 'nows'. The difficulty is then only in supposing the monads actually to act on each other.*

* As I have shown elsewhere, this conception bears upon the difficult question of the existence of evil; and is of great importance in ethics, for it supplies what was wanting in Leibnitz's theory.

CHAPTER XXI

TIME-WAVES

§ 1

WE have seen that change of time is accompanied by change of motion, that is force. Hence a variation of time means a propagated vibration of force; the points where the vacillations have effect being the seats of varying time. The 'time-wave' may in fact be regarded as a light-wave, in which the variations of time, either of acceleration or retardation, are the characteristic feature. Thus once more it is intelligible that the mind should become conscious of radiant energy.

The action of mind upon matter, as the result of the influence of mind on time (*vide* p. 297) is now seen to become manifest in the opposite or contrary action of radiation, as a wave of time-change, or variation thereof in space. If for the sake of argument we suppose consciousness to be a time-change phenomenon, then as time tends to equalize, a disturbance will be propagated through space. Such a relationship between cognition, conation, and affection has already been shown to be the result of the relativity theory. And now we find the converse, that radiation of certain frequencies ought, under certain circumstances, to excite consciousness. The wave of time-change is propagated from monad to monad in the order in which these are arranged in space, for this order is what constitutes space.

Whenever the silence of the universe is broken and disturbed by the message of a time-acceleration in space, this message may be witnessed as a ray of light, provided that the frequency of the vibration is of the right period. The almost established, though not necessarily pre-established, harmony of the universe of monads admits of such scintillations great or small throughout the universe. And the relativity idea of time is thus considerably extended, for not only is time, as relativity requires, the indicator of simultaneity in space, but the rate of propagation of a time-change constitutes the velocity of light itself. Change of time is therefore transmission of energy according to this view. And when God said 'Let there be light', He would appear to have disturbed the silence of the Universe; to have roused it up to consciousness as from a peaceful slumber, and awakened the mind to perception wherever His message reached and served to illumine the darkness of the firmament.

The variations in time are the means by which light is transmitted and in fact perceived. This wide applicability of the theory gives it unique value from a purely physical standpoint, whilst its bearing upon psychological phenomena renders it of far greater merit than that of the rival theories, which ignore the purely psychological and metaphysical sides of the question, for no other reason, so far as I have been able to conjecture, except that physicists have almost entirely ignored the very existence of mind in Nature, with the melancholy results of that intellectual bankruptcy of purely mechanical conceptions that the science of the latter half of the nineteenth century has so glaringly witnessed. The working out of a theory of the structure of the atom on such lines would form a separate branch of the subject for special investigation. But if the electron or the proton be regarded as the seat of such rapid vibrations that the oscillations could not be taken up by the surrounding medium, then its energy would be stored up within itself and outer radiations scattered or reflected. In my book *The Origin of Life* I ventured to put forward this hypothesis; but the time and the occasion were not then ripe enough for the suggestion to take root in the soil where it was sown. In my opinion this conception serves to bridge one of the great difficulties in Bohr's theory of the atom. For it is not known why an electron in falling from a higher to a lower ring of electrons should radiate energy, yet its energy is diminished. Wave-mechanics requires but does not give any explanation of the nature of these waves.

§ 2

The bugbear of the dissipation of energy, or the second law of thermodynamics, that the Universe is running down, entropy running up, ever seems to me to lead to little consequence. That universes are gradually transformed into atoms, atoms into electrons and protons, and these into quanta of energy, means a continual change from one scale of being to another and a smaller, though we know not if a better or a worse world. If these are again transformed into smaller, then the significance is only in the scale. If there is no standard of size, what then? If there is no infinitesimally small any more than infinitely large, then reality is in the changing process. And the change downwards in one direction rather than the other is but the physical or spatial concomitant of time, likewise a continual 'becoming' in one direction though in another dimension. But it will be said Planck's constant shows that there is a limit $= 6{\cdot}55 \times 10^{-27}$ erg/sec. What if this unit is itself a group, a universe of energy, conservative and self-

contained? If it could never disintegrate into the smallest, there yet is a smallest. Planck's constant is only a physical unit, like the atom or the electron of old. There are points, and I believe there are monads. Cantor's theory of the infinite applies to one, and Leibnitz's Monadology to the other. Points have no dimensions, neither have monads. Monads are mind units.

As there cannot be a nominative without a verb in a sentence that makes sense, so the monad if it is to make sense must be subject and predicate 'united in one', to make sense as a unit of reality. The monad is a Universe in itself. Its qualities are not described by $p \supset q$ but by $p \in q$. That is, its qualities are not related as *genus* and *species* or intersect as classes, but may be said to be united and to interpenetrate, only as ideas do. The distinction of 'types' ceases. We have a repetition of the process in the logical relationship of ideas. Thought as the mind-stuff of ideas or the mind-stuff of the monad is prior in its relationship with the material Universe of other monads by being the basis of their subsequent relations, which by selection, as we shall see, it is enabled to form. The member of a class and the class become one in a perfect unity. The one and the many become merged into an identity in differences, another world from the material, none other than the world of thought or reality. It is a reality independent of space or time.

This continuity between monads has curiously enough been the stumbling-block, as it were, has led to the abandonment of the conception of *pre-established harmony* in the Monadology. But all the monads in creation need not all be 'in tune'. They may, in fact, differ widely from each other in time, so long as they can be classified in groups or classes the members of which are in harmony with each other, though vibrating in time, some slightly ahead, others slightly behind the average for the group: the group forming the world or environment of each. It is the 'time-function' of each individual monad that gives it its true characteristics relatively to others in the world to which it belongs. For this function when analysed by Fourier's series will contain partials to which those of the group will respond. Space then becomes a particular grouping of monads beating time. But the properties of matter will depend upon the partials like the lines in the spectrum of an element. The continuity is in the actual simultaneity, as distinct from the relative time.

Each monad leads its own existence apart from every other. But

though not actually in harmony and beating time with any other, contrary to what Leibnitz supposed, some monads and in fact most monads in the Universe which constitutes our own are very nearly attuned. Those of our universe have some partials in common.

Physical time-differences between bodies is due to their relative motions and distances from each other. But the differences in time here contemplated between monads is yet of a different nature. Two monads relatively at rest may nevertheless not have the same *Now*. An effort of will may cause relative motion and a consequent difference of time between two bodies. And so a vibrating or changing tune in the monad, which we suppose to be the very essence of its life, would cause its time to vary relatively to that of other monads. It is from this conception that the theory here proposed is evolved. Bergson's clean-cut section of reality in time is seen to be a rippled surface, like tracings on a gramophone disk, which admit of revealing some tune of the majestic concert of creation to monads capable of responding to the variations of time-changes it performs. If we discard distinctions between great and small, an extension of Fourier's theorem yields obviously to the classification of monads in particular groups, that perform some rhythmic process from an acorn to an oak, an ovum to a man according to the type of its particular 'function'.

CHAPTER XXII

STIMULUS AND PERCEPTION

IN view of the above considerations it is possible to interpret physical terms in relation to perceptions. It is, however, possible to interpret in physical terms the very sensations themselves, so that the two become but different aspects of the same conceptions viewed from opposite extremes.

Thus in the familiar classical mechanics we have, if F = Force, m = mass, and a = acceleration, then

$$F = m.a.$$

Now since the acceleration varies inversely as the mass, for a given force, we may regard the reciprocal of the acceleration $\left(\frac{1}{a}\right)$ as the measure of the inertia or mass of the body.

Similarly for a given impulse P, if v be the velocity of the body of mass m, we have

$$P = m.v.$$

We may regard $\left(\frac{1}{v}\right)$ for a given impulse as a measure of the mass or inertia of the body.

The *momentum* is indicated by $m.v$. We may likewise call $\left(\frac{m}{v}\right)$ the *immomentum*.

Or if σ = the *slowness* of a body then, $\sigma = \frac{1}{v}$, and therefore the immomentum $= m.\sigma$.

Now the energy

$$= \int m.v.dv = \tfrac{1}{2}m.v^2.$$

We have similarly

$$\int m.\sigma d\sigma = \tfrac{1}{2}m.\sigma^2 = \tfrac{1}{2}m\left(\frac{1}{v}\right)^2.$$

Professor W. P. Montague of Columbia University, N.Y., has suggested that the term

$$\tfrac{1}{2}.m.\left(\frac{1}{v}\right)^2 = \tfrac{1}{2}.m.\sigma^2$$

might be called anergy, i.e. one half the mass multiplied by the square of the slowness.

Now as we increase v or $\frac{s}{t}$, we diminish $\sigma = \frac{1}{v} = \frac{t}{s}$, and vice versa, as we diminish v we increase σ, and likewise as the momentum, $m.v$, and the energy $\frac{1}{2}m.v^2$ are diminished, the immomentum, $m.\left(\frac{1}{v}\right)$ and the anenergy, as I would call it, $\frac{1}{2}.m.\left(\frac{1}{v}\right)^2$, or $m.\sigma$, and $\frac{1}{2}.m.\sigma^2$, respectively are increased.

Thus it would appear that if a state of rest is reached $v = 0$, therefore $\sigma = \frac{1}{v} = \infty$, i.e. infinity. And when the momentum and the kinetic energy are nil, the immomentum and the anenergy are infinite, and vice versa.

We may regard σ as the *duration*, since it represents the time required for a body to move through a unit of space, the velocity being the space traversed in a unit of time.

Hence if $\sigma = \infty$, the space is a point, and the duration is infinite. When the body has an infinite velocity its duration or slowness is *potential*, but when it is at rest, its motion is potential, and its duration or slowness *actual*.

Thus we must regard a body at rest as representing the opposite state corresponding to that of a body moving with an infinite velocity in which time is but a single instant.

Now since according to the theory of relativity v can never be greater than c, the velocity of light, it follows that σ can never be less than $\frac{1}{c}$, so that absolute slowness or absence of duration is impossible, and duration has therefore extension, both in space and time.

On the other hand, if v be relative and $v = 0$, then $\sigma = \infty$, so that slowness or duration would be infinite. An idea may therefore have infinite duration relatively, but it is impossible for it to have no duration. The distinction is of great importance in clear thinking. But in so far as according to the theory of relativity $v \neq 0$, absolutely then $\sigma \neq \infty$ absolutely.

In a paper read before the Aristotelian Society on 6 December, 1920, on 'Variation, Heredity, and Consciousness', Professor W. P. Montague suggests that this *infinite phase of slowness or duration* may be regarded as the 'common-stuff of all sensations', and that 'the critical points of zero and infinity through which the motion and the slowness respectively pass, afford the basis for that qualitative absoluteness and discontinuity that differentiates sensations from mere rates of change'.

He does not consider the limitations that the theory of relativity imposes upon these conceptions, but he deduces Fechner's interpretation of Weber's Law, that the intensity of a sensation is proportional to the logarithm of the intensity of the physiological impulse.

If we suppose $\sigma = \infty$ represents the *quality* of mere durational being common to all sensations, then if a wave of neural motion encounters resistance at the *synapse* each of the cross-sections in continuous series will be transformed accordingly into cross-sections of durational being which we may designate by δP.

Then the intensity of the sensation proportional to σ can be measured by the integral,

$$P = \int \mu \sigma dv = \mu \int \frac{dv}{v} = \mu \log v + C$$

(where μ is large for sound, and small for light) and C is a constant depending upon the initial values of P and v. This is Montague's conclusion.

That is if P = resultant sensation, and v = the stimulus, then

$$P = \mu \log v + C,$$

i.e. Weber's Law.

Therefore the intensity of the sensation increases as the logarithm of the stimulus rather than with the stimulus directly. That is, as the sensation increases in arithmetical progression the stimulus must increase in geometrical progression, which is Weber's well-known law.

Conversely if dQ = the impulse multiplied by the element of duration, we have

$$Q = \mu \int v d\sigma = \int \frac{d\sigma}{\sigma} = \mu \log \sigma + C',$$

$$= \mu \log \left(\frac{1}{v}\right) + C' = -\mu \log v + C'.$$

Therefore $P+Q = C+C' = K$, a constant.

If P is the actual sensation, and Q the potential or subconscious sensation, then *the sum of the actual and potential sensations is constant* for each psychological unit, or monad.

The actual sensation increases with the logarithm of the stimulus.

The potential sensation increases with the logarithm of the duration. This is thus another way of putting it.

The same result follows immediately from the definition of σ.

Since $\sigma = \frac{1}{v}$, we have $d\sigma = -\frac{1}{v^2} \cdot dv$,

$$\therefore \; v.d\sigma = -\frac{1}{v} \cdot dv = -\sigma.dv,$$

$$\therefore \; \int v.d\sigma + \int \sigma.dv = \text{constant},$$

$$\therefore \; P+Q = \text{constant}.$$

This conclusion is of importance in the theory of space proposed in the next chapter.

CHAPTER XXIII

THE NATURE OF SPACE

§ 1

THE manner in which any two monads can perceive the same idea or 'object' in space has always seemed to be an almost insuperable difficulty both in physical and metaphysical speculation. No clear idea, it would appear, has ever been formulated, even by those who have been most insistent on their own clearness, upon this extremely important but very obscure and intricate subject. The tendency in fact has ever been to slur over the difficulty, as soon as it had been reached, as though it were a mere insignificant obstacle unworthy of attention. But until the idea of space is clearly and properly conceived, little hope or chance exists of the relation of the phenomena of life to other phenomena of a purely physical nature, being recognized and understood.

Some attempts indeed appear to have been made by Herbart to deduce the idea of space mathematically. His deductions, however, have been submitted to all sorts of more or less severe strictures, but some parts of his work have stood the test of the most hostile and apparently destructive criticism, and may be regarded to-day as a part of the foundation of modern psychology.

Our concern at the moment is with the conception of space. Herbart's theory, though useful in the elucidation of memory, throws little light upon the actual difficulties considered here, or the theory which I now propose. It serves nevertheless as an illustration of previous attempts to tackle the problem as it should be handled with mathematical acumen and that scientific precision which results from it.

As a consequence of Leibnitz's theory of pre-established harmony between all monads, monads were supposed to be absolutely self-contained and incapable of acting on each other. This he himself held to be essential to his monadology. But the difficulties presented are even greater than those it endeavours to surmount. The admission of a plurality of monads, if we are to avoid mere solipsism, demands an interaction between them, which though perhaps not absolutely essential—for the monads may keep time like the separate players in an orchestra—yet their position in space is by no means intelligible on such a supposition, nor is a clear conception of space calculated to account for the facts of our experience on

such an hypothesis possible. Some form of interaction between monads must be postulated, and, as we shall see, it is the absence of such harmony that actually renders the conception of space possible, as well as the action of mind upon matter, and the interaction of matter with matter as fundamental facts of everyday experience.

The monad conceived by Leibnitz need occupy no space, and the velocity of light has no place in it. It is a unity of consciousness, existing somehow in what Kant might have called the 'noumenal world', of *things-in-themselves*. Each monad may have its own time, but only relatively to other monads, and there is no reason to suppose that they have all the same time. We have advanced much since Leibnitz's day in this respect, and provided that such monads are not all at the same time, as the modern theory of relativity requires, there is no difficulty in deducing a conception of space such as I here venture to propose.

This conception gives an entirely different perspective of Nature. We are bound to regard everything as the manifestation of a reality that is nowhere, but presented to each of us in his own particular world of 'space'. Each one being is related to all the rest, and views the rest from its own particular perspective.

If we dispense with the reality of space altogether on Berkeleyan lines, we are still encountered with the universal physical fact of the velocity of light, which requires that the separate monads must have different times. It is the classification of these in a gradual scale according to their 'time-differences' that introduces the idea of distance, and the Berkeleyan conception of space can be shown to fit in with that of mere time, for distance in 'space' becomes a consequence of 'time-differences' between monads throughout the infinite gradation of time-differences whatever the velocity of light. Some monads are 'before', some 'after', not only in the time of each, but of each relatively to each. Using polar coordinates r, θ, ϕ, t, we may regard r and t as practically equivalent. For $r = ktf(\theta, \phi)$, so that when θ and ϕ are given, r and t are proportional to each other, k being large and t small. This reduces the supposed four dimensions to three. The interpretation of the two quantities θ and ϕ is a question of psycho-physics.

If we regard θ and ϕ as 'factors' in a sensation, say of sight, then for a particular point in space we may likewise regard $i\theta$ and $i\phi$, where $i^2 = -1$, as the 'factors' of the sensation, provided that the sensation has two such 'factors', the idea of a space in three dimensions can be built up from it, omitting the constant k the correction for the velocity of light which may be taken as unity.

Let $$x_1 = \frac{t_1}{2}(e^{i\theta}+e^{-i\theta}), \quad y_1 = \frac{t_1}{2}\frac{(e^{i\theta}-e^{-i\theta})}{i},$$

whatever θ and ϕ might be, we obtain that $x_1^2+y_1^2 = t_1^2$, which is the theorem of Pythagoras in two dimensions. Similarly if

$$z_1 = \frac{t}{2}(e^{i\phi}+e^{-i\phi}), \quad \text{and} \quad t_1 = \frac{t}{2}\left(\frac{e^{i\phi}-e^{-i\phi}}{i}\right),$$

it follows that $x_1^2+y_1^2+z_1^2 = t^2$, which is the same theorem in three dimensions.

Moreover if m_0 be the percipient monad, and m_1 and m_2 any two other monads whose time-distances from m_0 are t_1 and t_2 respectively. And putting for simplicity for m_1, $\theta = 0$, and $\phi = 0$; and for m_2, $\theta = \alpha$, $\phi = 0$, so that m_0, m_1, and m_2, may be regarded as 'coplanar'. Then since the Theorem of Pythagoras has already been shown to hold, we have

$$\left[t_1-\frac{t_2}{2}(e^{i\alpha}+e^{-i\alpha})\right]^2-\frac{t_2^2}{4}(e^{i\alpha}-e^{-i\alpha})^2 = t_1^2-2t_1t_2\cos\alpha+t_2^2$$

as the well-known trigonometrical expression for the distance r_{12}^2 between m_1 and m_2.

We have shown that $P = \mu \log v+C$, and if P_0 be the standard value of a sensation for a stimulus v_0 then $P_0 = \mu \log v_0+C$, and consequently

$$P-P_0 = \mu \log\left(\frac{v}{v_0}\right), \quad \text{or} \quad \frac{P-P_0}{\mu} = \log\left(\frac{v}{v_0}\right), \quad -\frac{(P-P_0)}{\mu} = \log\left(\frac{\sigma}{\sigma_0}\right).$$

Hence $$e^{\frac{P-P_0}{\mu}}+e^{-\frac{P-P_0}{\mu}} = \frac{v}{v_0}+\frac{\sigma}{\sigma_0} = e^{i\theta}+e^{-i\theta},$$

where $$i\theta = \frac{P-P_0}{\mu}$$

$$e^{\frac{P-P_0}{\mu}}-e^{-\frac{P-P_0}{\mu}} = \frac{v}{v_0}-\frac{\sigma}{\sigma_0} = e^{i\theta}-e^{-i\theta}.$$

Now since these expressions are but the values of the trigonometrical functions of some angle which we may call θ, we have

$$\cos\theta = \frac{1}{2}\left(\frac{v}{v_0}+\frac{\sigma}{\sigma_0}\right), \quad \sin\theta = \frac{1}{2i}\left(\frac{v}{v_0}-\frac{\sigma}{\sigma_0}\right).$$

Hence for any stimulus v and its duration σ, if t be the distance of the point in time, as already explained, we have for the sum and difference of the stimulus and its duration two values $x = t\cos\theta$ and $y = t\sin\theta$ which obey the law that $x^2+y^2 = t^2$, from which a sense of direction is by experience inferred, though not of course

with any immediate consciousness of its full geometrical signification.

The same considerations would apply to a third dimension.

The interval t is too small to be perceived at ordinary distances but the monad places the idea as though it were in 'space'.

If the sense of sight be supposed to give an idea only of two dimensions, as in a picture, then the sense of touch, which we may regard as totally distinct, gives the idea of extension. For if v' be the stimulus through other sets of nerves in this case, and Q the sensation, and $i\phi = \frac{Q-Q_0}{\mu}$, we obtain

$$z = t\frac{(e^{i\phi}+e^{-i\phi})}{2}, \quad \text{and} \quad t_1 = t\frac{(e^{i\phi}-e^{-i\phi})}{2i},$$

so that $$x^2+y^2+z^2 = t^2.$$

P and Q must be sensations of two different 'kinds' but simultaneous, one of sight and the other of touch, the relation between t_1 and t being an experiential one.

It might be said that an additional sensation of a different kind such as colour, hearing, taste, or smell would give an additional sense of dimension, but this is not so, owing to the undeveloped nature of these faculties, though there may be creatures to whom such senses might afford an extra dimension of space, in a manner to us quite incomprehensible. The point is that the combination of any two senses with that of time, if the latter represents distance, gives the sense of extension.

Peano's geometry is derived entirely from the conception of classes, and worked out by his system of symbolic logic, upon the same lines as his arithmetic. There is no attempt, however, to build up the idea of space from that of time and physiological sensation, but it is clear that a synthesis such as is here proposed is feasible on his lines, the theorem of Pythagoras being taken as the basis of geometrical space in three dimensions.

It can of course be urged that the sum and the difference of a quantity and its reciprocal can always be expressed as proportional to the cosine and the sine of some quantity. But this does not detract, nay rather does it confirm, the contention that when a stimulus is combined with the duration, two orders of sensations are thereby produced, according as the duration is added or subtracted from it. Thus $\frac{v}{v_0}$ will always depend upon the direction, and the duration will likewise do so. Conversely the direction will

depend upon both. There is no going behind this fact whatever direction might mean. It is only experience that teaches us what space means as appearance. Of the sum of the stimulus and the duration we may perhaps form some conception, but the difference, what does it mean? Yet as we have seen (p. 161) the negative in fact corresponds to the reality, though no conception can be formed of it, except as the resultant effect which experience confirms or affirms. The negative of time is in the subconscious, and as an intuitive process helps to produce the resultant effect. Hence the synthesis of two effects apparently derived from two different categories yields the resultant effect known as 'space', itself derived partly from an intuition of negative time, the very counterpart of that time which is unidirectional but whose opposite, like the images in a mirror lies hidden in the subconscious. Just as the mind can distinguish colours of different frequencies, so in virtue of the psychological constants κ and μ, P_0 and Q_0, it can distinguish distance from direction.

The use of hyperbolic functions is particularly significant in this respect, for the interpretation of θ and ϕ, physiologically giving the idea of direction and location as the result of habit or custom from the stimulus of a sensation, leads directly to the idea of space.

The sense of permanent position or place in space is thus acquired. Just as distance from the percipient monad or subject is given by the difference in time, the actual velocity of light 3×10^{10} cm. sec. has nothing to do with it but only the sensation P, so direction appears to be determined by a function of the sensation, i.e. that of the stimulus, independently of the magnitude or the distance. For it has been shown that whatever be the intensity of the sensation or the impulse of the stimulus to which it is due, given certain initial conditions, the sum of this magnitude $\frac{v}{v_0}$ and its reciprocal $\frac{\sigma}{\sigma_0}$ is constant, so that the sense of θ and ϕ, whatever these might be, is independent of the intensity of the sensation. Hence from the fact that the sensation and its reciprocal are constant, the idea of space and of something permanent and independent of the intensity of the sensation is built up. Direction is thus a sort of hyperbolic function of the sensation expressible as the mere transformation of *coordinates*. Thus distance and direction, as well as position resulting therefrom, become functions of duration and sensation, since we have to deal with the sum and the difference of the stimulus and its reciprocal, that is in the latter

case, the slowness of the stimulation, namely, the duration. If the time of a monad differs from that of other monads, as we have supposed, it may also vary relatively to these round a certain average, as a sort of nutation effect. This in fact is what we should expect to be actually the case. Hence the variations in time of, or at, a particular monad would be periodic, and, by Fourier's Theorem, harmonic, as a function of the time at that point. It can be represented by a function of the sum or the difference of two exponential functions, namely the cosine and the sine of the time, the hyperbolic functions of the type just considered. Thus the three-dimensional nature of space, or four-dimensional 'space-time continuum' would appear to be accounted for, since three, and therefore four, different conceptions are involved, each of which represents a distinct and definite idea admitting of interpretation 'as though' space existed, the ideas of direction and distance, in other words of locality, following immediately from them.

If light is a scintillation of time difference, the immense velocity at which it is propagated does not affect the perception of the scintillation. Distance in space becomes a difference in time, which cannot be perceived except in stellar 'space' but for the constants κ and μ; the monads being in reality non-spatial, or if we prefer to put it so, higgledy-piggledy, are placed by the mind of the individual monad in the order or class of 'time-differences in relation to itself', that is, the order in which the time-wave of acceleration, or accelerated time-wave, takes place. And this order is what it designates as the order in space, for it is what is imagined to be the distance. The monads which are 'farthest away' are merely those that would take longest to be influenced by a time-change phenomenon. *Space is thus annihilated so far as distance is concerned.*

§ 2

Further, the more a monad is out of tune with the percipient monad or subject, the more remote is it also in 'space', for the longer does it take a message to reach it, or to come from it. Direction is then the indicator of the path of least time, as distance is the time-difference along that path. This is the line of quickest communication, 'a geodesic' on an imaginary surface. The manner in which one monad affects another being through the infinitesimally small difference in time between it and certain others of the infinite host, with which it may be regarded as being in immediate 'contact' or continuity in time. The number of monads is assumed to be practically infinite, but there is for any particular monad a

class of monads whose time-difference from it is infinitesimally small. Hence all monads may be considered as related to each other, both in time and 'space', and to admit of being classified by each according to the order of their time-difference relating to it.

Thus it can be shown that for all purposes concerned it makes no difference whatever if space be annihilated, provided that the monads which are perfectly at random *en masse* in relation to each other are susceptible of selection in groups that respond to the harmonic partials of any particular *time-difference* or displacement constituting a vibration of time-difference round a certain average. For obviously any time-change disturbance that is taken up by any other monad differing by an infinitesimally small amount from it will in turn affect another and another *ad infinitum* so that the disturbance will be propagated in time to all monads, even to those differing from it to an infinite extent in time. A little stretch of imagination will enable the reader to conceive that if time-difference be regarded as the true measure of distance, all monads can be grouped in classes that might be pictured by any individual monad as spatially related to itself. The distance of any particular monad from any other is represented in its own class by this time-difference, and its position by a function in which this quantity is a factor, so that the two are proportional to each other.

Thus we appear to surmount a difficulty that Kant encountered but in his day never overcame, namely that all phenomena in space are also in time, but those in time are not necessarily in space. For space, as we have seen, may be conceived as a property of time.*

When an electron is displaced so that its position within the atom falls from higher to lower potential, its time is likewise altered. A wave transmitting this change is sent 'outwards' throughout all 'space'. This 'time-change' phenomenon may be regarded, as we have seen, as equivalent to an electric displacement in the classical physics. The manner of response of monad to monad depending upon the readiness with which the 'time-change wave' is taken up, for that a physical connexion between monads adjacent in time does exist is hardly to be questioned, as long as the phenomenon of the transmission of light is admitted to take place.

That light appears to take time to be propagated is due to the fact that the time-change is transmitted by a gradual though rapid

* The late Professor Swift Paine Johnston of Trinity College, Dublin, proposed this problem to me more than twenty years ago, and I have since had occasion to evolve the solution of the problem here presented. It places space in the 'Category' of time, and renders it a consequence of Weber's Law, and Leibnitz's Monadology.

process, owing to the continuity in time between monad and monad, resulting from the infinitesimally small difference in time between some pair in every case amongst the infinite host. The classes overlap. The effect then is from those that are nearest in time to those that are farther away; and distance thus becomes a difference in time owing to the fact that such time-changes take time to be transmitted.

§ 3

The Theory of Relativity clearly would apply to the space so conceived as to the ordinary space of our familiar notions.

The time t at a point m_0 and the time t' at m' are still related by the Lorenz equation

$$t' = \beta\left(t - \frac{vd}{c^2}\right),$$

where v is the supposed relative velocity, and

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}},$$

where c = velocity of light, t' being the time at m' relatively to axes at m' moving with the velocity v outwards from m_0. The time we have been considering is that at the various classes of monads equidistant from m_0 in time, imagined as lying in spheres around m_0. If t' is the time at any such 'sphere' or class m' and t the corresponding time at m_0, then $t' = \beta\left(t - \frac{vd}{c^2}\right)$. Hence when $v = c$ we get $t' = 0$, since $d = ct$.

We have chosen in our theory of space the hypothesis, making the distance proportional to the time. If a ray of light started from m_0 at noon, then at 1 p.m. it would be at a region where the time would be noon, and time for the ray would be timeless, but if we could perceive the order from m_0 we should admit that relatively to m_0 all other monads are behind it if they take their time from it, and vice versa before it otherwise.

Hence some can be before and others after only if there is an absolute time. But that there must be such an absolute time does not alter the fact that with respect to each monad the time of every other is merely relative. There is an absolute 'now', but there is no absolute measure of time. Such monads as make up the universe with which we are concerned would appear to range around a certain average time, the absolute being indeterminate though absolute, as a reality: else history and evolution would not be possible. We have supposed the variations from this absolute

or the average not to be constant for each monad, but to vary within certain small limits, so that a time-variation is a characteristic of each monad.

§ 4

All monads which have the same such frequencies or partials in Fourier's series are supposed to respond. Hence *however they might differ in time, or frequency,* provided they had common partials in the Fourier series of their respective variations they would be in harmony and respond to each other.

Such a system or class of monads, if at approximately the same time, will constitute a group possessing different qualities, and yet be in approximately the same 'place' in 'space'.

This simultaneous presence would constitute the distinctive characteristics of matter as also the monads forming a group of ideas or thoughts. The latter are not in space and need no time propagation. Moreover, when the vibration of any particular monad is resolvable into a very great number of partials by Fourier's series, then it can respond to a corresponding variety of monads throughout its universe, spatial and intellectual. The larger the number of partials or harmonics the greater the number of monads with which it will be in 'tune'. And still more, a group of monads closely allied in time will obviously admit of the response to chords of a most complex nature, with which those of music as adapted to the human ear can but little vie. In diversity and complexity, in range of period and freedom of play, matter acquires a potentiality with which by nature it appears to be endowed. Nay more, since from the nature of the idea as here conceived as itself a monad of permanent reality, all thoughts, all passions, all desires in the individual mind of the monad that constitutes myself and each of us in his own sphere of consciousness are but the response to the infinite host throughout the boundless realm of material or idealistic creation; and beyond it when the range is not controlled by the guiding elements that reason has prescribed as the eternal laws of rational thought. Neither Schopenhauer nor Bergson has been able to present by the mere influence of will the communion of the self with reason and reality, and the appearance as if from nowhere of types or living archetypes corresponding to the eternal Idea of the Logos.

§ 5

The tendency is for monads so influenced to be forced to equalize their time where there is a slight difference of phase, almost of an infinitesimal nature, between them. Hence there

would be a tendency for the 'distance' between such to diminish. But whatever the variations in this respect, there is always a tendency for an ideal wave of time-change to increase or diminish the time of any monad it affects. When an electron changes its position in the atom by $-dx'$ then dt' is positive, since, as in p. 324,

$$dt' = \frac{\beta^2(-x')dv}{c^2}.$$

Hence there is a slowing down wave in time. But the time of the electron will be behind that of other monads, relatively to its previous state. The tendency to equalize will therefore tend to diminish that difference, i.e. to diminish the distance or in other words to attract. From the equation in p. 331, it would appear that the rate of change of time in time Δt, if t is proportional to the distance, is inversely proportional to it, i.e. to the distance in time. Hence the differential of t' with respect to t, for there is a retardation in dt which is positive as in the case of clocks going slower, is the time-difference with respect to the time, and is inversely as the square of the time, i.e. of the imagined distance. Thus for practical purposes we may regard the Newtonian Law of the inverse square as holding to a first approximation. The full expression for the Law of Gravitation is the same as in the familiar notions of space, so that the Einstein Theory of Relativity likewise holds good.

In either case the expression $dx^2+dy^2+dz^2-dt^2$ remains invariant, i.e. dr^2-dt^2, or since $r = \mu t$, $(\mu^2-1)dt^2$. That is, the element of time is constant, being in fact the only interval in question.

In this case μ is obviously the same as c the velocity of light.

The monadology of Leibnitz no less than the modern quantum theory enters here, although it must not be assumed that I have in any way endeavoured to adhere to his conceptions in their modern aspects. We have moved on rapidly in thought since his day, particularly during the past decade, and to follow him explicitly, as the Schoolmen did Aristotle almost as an oracle, would be absurd, in the light of modern knowledge. The conception of space I have ventured to put forward here differs largely from that in the monadology, though not fundamentally so, implying as it does the idea of the monad as its basis. The present theory, however, must be regarded as a synthesis and emergence on a different plane from the constituents out of which it has evolved.

The distinction between matter and form, or accidents and substance in the old Scholastic and Aristotelian philosophy, becomes one of intrinsic value. For although at first sight there would be no such substance, as indeed Berkeley supposed, since *esse est percipi*, yet monads do exist, even when not being perceived by me, and are either being perceived by the Absolute monad, or are self-perceiving entities. Moreover, though 'substance' in the Aristotelian sense has according to my theory no very definite meaning, the characteristic property that the monad possesses of picking out the partials in its Fourier components, enables it to perceive as in a particular place all the harmonics of a particular time displacement.

So far we have omitted to consider the bearing of these conceptions upon the recently developed wave-mechanics, or conversely the bearing of the latter on the former. The pulsations ψ assumed in the wave-mechanics, whose nature is not even conjectured, can be supposed in view of these considerations to be time-variations or pulsations, and all that has hitherto been said could apply equally well to this later theory, freeing it somewhat from that incomprehensible mystery that it otherwise possesses. The ideas proposed by de Broglie, Heisenberg, Schrödinger and Dirac, like the earlier but now classical physics, admit of interpretation in accordance with this idea of time.

§ 6

The underlying hypothesis in the wave-mechanics is that 'Every "mass particle" at rest is the centre of a pulsation throughout space. Or, more definitely, in a (space-time) system of reference K_0, with coordinates x_0, y_0, z_0, t_0, in which a "particle" is at rest, this particle is to be regarded merely as a "singularity" of a pulsation which for any given time is in the same phase throughout space.' (See H. F. Biggs, *Wave-Mechanics*, Oxford University Press.) 'The "region occupied by the particle" is merely the region where a set of ψ-waves, varying continuously in direction and in frequency over a small range, reinforce each other to form a *wave-group* travelling with what we call "the velocity of the particle". Since waves of different frequency travel with different velocity we have the phenomenon of a dispersing medium, in which, as is well known, the group, or region of reinforcement, has a velocity different from that of the phase' (loc. cit., p. 22).

The curious thing is that ψ can be represented as e^P and its conjugate $\bar{\psi}$ as e^{-P}, $\psi.\bar{\psi}$, being taken as the electric density. We take x and y as proportional to $(\psi+\bar{\psi})$ and $(\psi-\bar{\psi})$, and the concep-

tion of space itself is reduced to a function of time. The functions giving the values of x, y, z, are spherical harmonics, and the treatment of the problems involved reduces to those of now familiar physics of wave-mechanics. A full analysis on the lines indicated will, it is hoped, be given in due course. But for the present it is not so much the physical as the psychological aspect in relation to our ideas of space that interests us most.

The essential point in the idea of space I have formulated is the conception of 'time-change'. The acceleration or retardation of time of a given monad, relatively to all other monads.

In Einstein's quadratic function for the 'interval', ds^2, namely $\sum g_{mn} dx_m dx_n (m, n = 1, 2, 3, 4)$, which has the same value however the coordinates may be altered, a change in the field of force by say an act of volition can be regarded as producing a change in some of the constants g_{mn}, so as to alter the quadratic form. This indeed is then what happens when under the influence of will, a body is put into motion. The interval is altered. Moreover, we suppose that there is a pulsating value for ds, and that as in the simpler case of $ds^2 = (c^2 - 1)dt^2$, the change is in dt itself.

A. A. Robb (loc. cit.) says:

> If, with Einstein, one takes a particular quadratic form as corresponding to a particular gravitation field, we might suppose that a change in the gravitation field produced by something else than gravitation (say, for instance, a change which was initiated by an act of will on the part of a living creature) corresponds to a change in the quadratic form. If it be permissible to take such change as the influence which gives the *before* and *after* relations their physical significance, instead of using optical phenomena for this purpose, then we might suppose that the *before* and *after* relations, as formulated, hold in some sense whether matter be present or not, but that it is only in the absence of appreciable quantities of matter that the postulates can be interpreted strictly optically.

This shows that Robb considers an absolute space and time not incompatible with these conceptions of relativity.

The substitution of t for r in Schwarzschild's well-known equation simplifies its solution considerably. This is a merely familiar algebraic analysis.

The mystic formula $qp - pq = \iota h/2\pi$ in wave-mechanics, where q are coordinates and p momenta, $\iota = \sqrt{-1}$ and h the Planck constant or atom of action $= 6.55 \times 10^{-27}$ erg. secs., requires that $qp \neq pq$. Schrödinger regards p as an operator, Born and Jordan as a *matrix*. But Dirac takes the view that p's are mere symbols, q members which do not conform with the rules or laws of ordinary Algebra, for they are not commutative. I suggest here that they are

transfinite numbers. Infinite numbers are not commutative and depend upon their order. Moreover, the difference between two infinite numbers may be finite. pq and qp are quantities of action. Although made up of infinite numbers they can be treated as entities that obey other laws than those of ordinary arithmetic. The entity ψ which is supposed to oscillate in wave-mechanics may be represented as a probability. It is the chance that a particle or electron would be in a particular region. If it is concentrated within a very limited area, it becomes almost certain to include an electron. As the monads are points, the concentration of monads within a very small 'stormy area' may correspond to ψ, and the waves themselves to 'time-waves'. We likewise supposed that 'time-waves' are continuously passing throughout the universe of monads as they are not in harmony and ever oscillating, events never being simultaneous between them, contrary to Leibnitz's pre-established harmony. These time scintillations thus afford a basis for wave-mechanism and in the manner indicated later of the phenomenon of life.

According to the logical theory I put forward, $xy \neq yx$ at different times, for instance.

Thus $x_{t_1} y_{t_2} \neq x_{t_2} y_{t_1}$ or in the case of $x^2 \neq x$ the logical proposition reduces to a Hegelian contradiction. Moreover, where there is continuity, or where we have to deal with infinite numbers, $xy \neq yx$.

§ 7

The differences between monads are sufficient to account for their characteristic properties. All their physical properties at least can be represented by their time 'function'. The particular mode of time oscillation of any monad is unique and can be regarded as different from that of every other. It can be represented by a curve resolvable into Fourier's series of components. And accordingly its relations to other monads become dependent upon these. The monad must be regarded as the seat of an extremely complicated mode of time variation relatively to the universal monad, or the average monad in time.

When the 'time-function' of a monad is resolved into its Fourier components, it is easy to see that some other monads having the same components or partials in common will accordingly be attracted or repelled by it. That in fact its influence in the selection of partners will depend largely if not entirely upon these partials.

One can readily understand that with the whole gamut from + infinity to − infinity, there is a grand opening for the per-

formance of the most majestic concert, such in fact as creation itself performs.

The most complex combinations characterize the peculiarities of the monad, and enable it in certain circumstances to pick out whatever types satisfy its own propensities. Thus, acting as a nuclear substance, it would at once be capable of drawing as it were unto itself the monads that suit its apparently predetermined ends, without any purposeful ultra-mundane influence other than the individual movement which distinguishes its individuality apart from other entities. Here we have the distinctive and purely mechanical operation which results from the mere difference in time from the absolute monad, varying from the absolute time in a definite way and imparting to the monad so displaced the qualities which, devoid of this difference, it would not possess; but which when once endowed with these would enable it to gather to itself the elements in chemical aggregations, and group them into whatever varied forms its own individual stamp of function might impart to them.

By a suitable arrangement of partials the groupings such as are witnessed in the living cell become possible. For the selection of partials is a matter of synchrony, and when the adjoining molecules respond to the vibrations of retarded or accelerated time, the stronger acting as a nucleus gathers unto itself the synchronizing elements, and evolves 'around' it the groups best suited to its purpose, for purpose it may well now be called, guiding as it were the development of myriads of atoms and molecules along the tortuous paths of living substance in its activity and growth. This grouping inevitably brings about the self-adjustment of the 'genes', 'determinants', or 'gametes', and causes them to form into array like the individual soldiers in review or on the field of action ready to pursue and to perform the function of their individual parts in the organization of the whole; and as such the intricate 'structure' or rather complex series of harmonic partials of the nucleus, in almost endless array, form, like the continuous decimals of our earlier illustration, the cellular structures endowed with power for the actual realization of some purpose. All this must be considered as a separate and distinct branch of the subject. It is sufficient for the present to point out its bearing upon the general theory.

§ 8

A ray of light is the simplest manifestation of this process. When a displacement is effected by the disturbance of the difference of phase or of period of an individual monad, the wave so propagated

serves to arouse such difference as it possesses in all others capable of responding to its call, as long as the energy suffices. Sir Leonard Hill's suggestion that light is nothing else than vitamine D, is of particular interest in this respect. The action of light, however, on colloid bodies, such as glass was shown in my book on *The Origin of Life* (1906). This was considered at the time to be a fantastic notion, but 'history will have to decide such questions'; and we cannot act as our own judges—authors and critics alike, all must bow, not to the prejudiced views of yesterday or to-day, but to the unbiased verdict of the years to come.

When a ray of light enters the eye we have, if x_1 and x_2 be two magnitudes and if $x^R = e^y$, $y = R \log x_e$,

and $$\frac{dy}{dx} = \frac{R}{x}, \qquad \frac{d^2y}{dx^2} = \frac{-R}{x^2},$$

and further, $$\left(\frac{d^2y}{dx^2}\right)_{x_1} - \left(\frac{d^2y}{dx^2}\right)_{x_2} = \left(\frac{1}{x_2^2} - \frac{1}{x_1^2}\right)R.$$

The latter is the Rydberg expression for the frequency of the spectral lines, when x_1 and x_2 are integers, and the certain constant R is introduced as a factor of the quantity in brackets, equal to nh, where n is the frequency of light and h the Planck constant. In other words

$$\left(\frac{d^2y}{dx^2}\right)_{x_1} - \left(\frac{d^2y}{dx^2}\right)_{x_2} = nh.$$

According to Weber's law of perception, if P be a sensation and v the stimulus, $P = \log . v +$ constant. Or if P_1 and P_2 be two sensations, and v_1 and v_2 the corresponding stimuli, then

$$\left(\frac{d\omega}{dv}\right)_{v_1} - \left(\frac{d\omega}{dv}\right)_{v_2} = \left(\frac{1}{v_1} - \frac{1}{v_2}\right),$$

and $$\left(\frac{d^2\omega}{dv^2}\right)_{v_2} - \left(\frac{d_2\omega}{dv^2}\right)_{v_1} = \left(\frac{1}{v_2^2}\right) - \left(\frac{1}{v_1^2}\right).$$

Hence the difference of the second differentials for v_2 and v_1 of the sensations follows the Rydberg law of spectral frequencies of stimuli, when the stimulus is discontinuous.

As we have seen, the reciprocal of the stimulus v may be regarded as a measure of the duration, so that the change of sensation is proportional to the duration, and the second differential of the sensation is proportional to the variation in the duration. *This variation of duration is an acceleration or retardation of time*; a group wave.

Now, wherever an acceleration or retardation of time occurs a

corresponding flow of energy ensues, if the *action* of the system remains constant as it would during an event, since

$$Energy \times Time = Action.$$

The same principle applies to Bohr's atom, since the displacement of an electron from one orbit to another means a change in its time and therefore its energy. So that if we take the simple case of two bodies relatively in motion whose times are t and t' respectively

$$t = \beta\left(t' + \frac{vx}{c^2}\right),$$

and when t' is suddenly altered, t must suffer a corresponding change. Hence a flow of energy takes place between them, the direction depending upon whether the time is increased or diminished. It follows that when an electron is displaced in the atom, a flow of energy in the form of radiation extends throughout space. Conversely, when there is a flow of energy there is a corresponding variation of time, which under certain circumstances can give rise to a change of the sense of duration, and a corresponding sensation.

§ 9

It appears also that what was meant by light being a 'time-wave' is equivalent to the familiar notion of an energy wave, provided that the action is an event when the time-change remains constant.

Hence also an atom may be regarded as a seat of motion from which no energy proceeds except when a displacement of its constituent particles occurs.

It may be said that the familiar conception of transmission of energy is far simpler than that of a variable time, not as we have seen from a psychological point of view.

There is a state in which a change of time by acceleration or retardation does not affect the time of bodies surrounding the particular seat of the disturbance, and this is plainly manifest when the condition of affairs is as portrayed on p. 328 et seq., or else the vibrations are too rapid to be propagated, as on p. 301. Obviously, moreover, if

$$dt' = -d\left(\frac{vx}{c^2}\right),$$

we have in general

$$t' = \beta\left(t - \frac{cx}{c^2}\right), \quad x' = \beta(x - vt),$$

$$t = \beta\left(t' + \frac{vx}{c^2}\right), \quad x = \beta(x' + vt),$$

and

$$dt' = d\beta\left(t-\frac{vx}{c^2}\right)+\beta\left(-\frac{x}{c^2}dv\right), \quad \text{where} \quad \beta = 1\Big/\left(1-\frac{v^2}{c^2}\right),$$

so that

$$d\beta = \frac{\frac{v}{c^2}dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \beta^3\frac{vdv}{c^2}.$$

Hence it can easily be seen that

$$dt' = \beta^2\frac{dv}{c^2}(-x') = -\beta^3\frac{dv}{c^2}(x-vt).$$

Similarly

$$dx' = \beta^2 dv(-t') = -\beta^3 dv\left(t-\frac{vx}{c^2}\right), \quad \text{or} \quad x'dx' = c^2t'dt',$$

i.e. $x'^2 = c^2t'^2+\text{Constant}$. Therefore $dt' = 0$, if $dv = 0$, or if $x-vt = 0$. In case $dx' = 0$ the motion within the atom is steady. That is if the distance is constant between electron and proton, $dt' = 0$, no radiation takes place.

The conception of Action = (Energy × Time) is thus a particular case of the general principle considered above. As the one increases the other diminishes, i.e. the acceleration or retardation in time is accompanied by the flow of energy. When time does not suffer acceleration or retardation, and the action is constant, the energy is constant during an event.

There is, however, a danger of misapprehension when the statement is made in familiar language in this form. Action is no doubt of the dimension of energy multiplied by time. But, for the element of action, the time is an infinitesimally small interval. If A = Action, Q = Energy, and t = Time,

$$A = \int Q\,dt.$$

Here the action is never constant for successive events if the energy is constant. But if for any value of t the time is altered to t', as in the theory of relativity, where the action remains constant during an event, then $\delta A = Q\delta t = Q'\delta t'$. The action at a point remains constant during an event. The curve along which the action remains constant in the 'space-time continuum' is called a *Geodesic*. It is not necessarily the path of least action. But since $\frac{dA}{dt} = Q$, then $\frac{d^2A}{dt^2} = \frac{dQ}{dt} = 0$ *when* Q *is constant*, and therefore A is then a minimum.

But when the time is suddenly altered and the action remains constant the energy is altered. If Q becomes Q' when t is altered to t', then since $t' = \beta\left(t - \frac{vx}{c^2}\right)$, we have $\frac{Qt}{Q'} = \beta\left(t - \frac{vx}{c^2}\right)$.

We have seen from equations (p. 324) that $dt' = \frac{-\beta^2 dv(-x')}{c^2}$,

and $$dx' = \beta^2 dv(-t') = -\beta^2 dv\left(t - \frac{vx}{c^2}\right).$$

Hence since $Qdt + tdQ = 0$, or $\frac{dQ}{dt} = -\frac{Q}{t}$, a diminution in t means an increase of energy.

If the action is altered during the event, then

$$dA = Qdt + tdQ = \frac{Q\beta^2 dv(-x')}{c^2} + tdQ.$$

In either case there is a change of velocity and therefore a change of energy.

The bearing of this on the general problem is indicated in the sequel.

CHAPTER XXIV

RELATIVITY AND THE MONAD

NOW if there are two consecutive points with the same time, and the time of one is suddenly altered, if the new time is to prevail there must be a wave transmitted forwards to readjust the system in order that the relations with the time at the origin should hold.

If we suppose, as it is not unreasonable to do, that the time of a monad varies harmonically from the average corresponding to actual position relatively to surrounding monads, it will experience a force that varies likewise in a harmonic manner. Thus, as we have seen when t' becomes altered by dt',

$$dt' = \frac{\beta^2 dv(-x')}{c^2}.$$

Consequently, if t' vary harmonically,

$$t' = t(1+A\sin wt),\ \therefore\ dt' = Aw\cos wtdt.$$

Hence
$$\frac{-\beta^2 dvx'}{c^2} = Aw\cos wtdt,$$

and
$$\frac{dv}{dt} = \frac{-Awc^2\cos wt}{\beta^2 x'} = K\left(1-\frac{w^2t^2}{\lfloor 2}\right).$$

It follows that if there be two such vibrating monads, or rather monads with vibrating times, the difference of the apparent forces with which they would be attracted towards the origin would be

$$K_1(n_1^2-n^2).$$

The differences are then proportional to the difference of the squares of frequencies.

The change in the position of an electron is accompanied by a change in its time. And a change in this with reference to any system of coordinates must mean a change with reference to every other, but the message can be transmitted only with the velocity of light. And if this change of time is an oscillating one the wave of transmission is likewise oscillating. It is the transmission of this change as a group that constitutes light on the lines here indicated.

A message can thus be transmitted to those immediately almost in tune or in harmony with it, since then the kind of space we have conceived would be possible. Distance being the interval of time, the space coordinates would no longer be independent variables, but functions of the time.

The propagation of time-change in space is the fundamental physical phenomenon. Light is a disturbance of the space-time relations of the Universe.

To the equation of identity $x = x^n$ we should add the further fact that $x^n = f(t_n)$. For the precise nature of the identity in the sense in which the term has been employed, the power of x may be a function of the time. Hence the differential coefficients

$$\frac{dx}{dt}, \quad \frac{d^2x}{dt^2}, \ldots, \quad \frac{d^nx}{dt^n}$$

become full of logical meaning. But if $x^t = x$, then as before the only real value of x is unity. Nevertheless, the imaginary roots of unity, usually denoted by ω, can be dealt with *as if* they were real—'als ob'; that is, if a class changes in time it can do so uniformly or with an acceleration in various infinitesimal degrees.

Although it may be necessary to descend to the nth differential, this would not invalidate the result.* Differences leading to variations may arise. Variations in one scale lead to mutations in a higher. And conversely, as it would well seem, the mutations in any particular scale are due to differentials of the first order giving rise to variations in the next lower scale, and so on.

The most important point in this respect now is as regards the variations of time itself at any particular place, and the wave of propagation of this time-change in space. $\frac{dt}{dx}$, what we have called the 'slowness', now represents the variation of time along space, and $\frac{d^2t}{dx^2}$ is the variation of this variation or acceleration of time in space, whether in progression through space or in the mere change of dimensional units. Now, if we regard light as such an acceleration of time, a world of new possibilities is opened up before us. As has already been shown in p. 326, a place where the time varies harmonically is a centre of force, the magnitude of which is proportional to the frequency of the vibration.

Let us consider two bodies A and B moving relatively to each other, and suppose the motion to be regarded with respect to axes K and K' respectively. Then, as we have seen, the Lorenz transformations give, if x and t stand for K and x' and t' for K' respectively,

$$x' = \beta(x - vt), \quad \text{and} \quad t' = \beta\left(t - \frac{vx}{c^2}\right), \quad \text{where} \quad 1/\beta = \sqrt{\left(1 - \frac{v^2}{c^2}\right)}.$$

* *Vide* T. Tovani, *Proc. Aristotelian Soc.* (1921).

Therefore $dt' = \beta dt - \beta \frac{v}{c^2} dx$, v being constant, where dx is negative.

Then $dt' = dt$, if $\frac{dx}{dt} = \frac{\beta - 1}{\beta} \frac{c^2}{v} = \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) \frac{c^2}{v}$.

If $v = 0$, this becomes $\frac{0}{0}$, but its actual value is $= 0$, as can easily be seen by differentiating the numerator and denominator, and making $v = 0$.

Again, if $dt' = dt$, and $v \frac{dx}{dt} = \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) c^2$, the clocks in the two systems would appear to go at the same rate. But they would not be simultaneous.

The further condition of simultaneity is of course $t' = t$, so that

$$(\beta - 1)\, t - \beta \frac{v}{c^2} x = 0.$$

$$\therefore \quad \frac{v.x}{t} = \left(\frac{\beta - 1}{\beta}\right) c^2 = \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) c^2 = v \frac{dx}{dt}.$$

$$\therefore \quad \frac{x}{t} = \frac{dx}{dt}.$$

But $$x' = \beta(x - vt) = \beta \left(\frac{x}{t} - v\right) t,$$

$$\therefore \quad \frac{dx'}{dt} = \beta \left(\frac{dx}{dt} - v\right),$$

if the velocity v is uniform to a first approximation.

If $v = 0$, then since $\beta = 1$ it follows that $\frac{dx'}{dt} = \frac{dx}{dt}$. (1)

If $v = c$ then $\beta = \infty$, and $\frac{dx'}{dt} = -\infty$. (2)

Whereas when $v = \frac{dx}{dt}$ we simply have $\frac{dx'}{dt} = 0$. (3)

If $v = \frac{dx'}{dt}$, then $v = \beta \left(\frac{dx}{dt} - v\right)$. $\therefore \quad \frac{dx}{dt} = \left(\frac{1 + \beta}{\beta}\right) v$. (4)

That is $$\frac{dx}{dt} = \left(1 + \sqrt{1 - \frac{v^2}{c^2}}\right) v.$$

This is the velocity of the body B relatively to the K axis, while its time relatively to both the K and K' axes is the same, i.e. the bodies A and B are simultaneous.

In general $\frac{dx'}{dt} = \frac{dx'dt'}{dt'dt}$, and since $dt' = dt$,

$$\frac{dx'}{dt} = \frac{dx'}{dt'}.$$

Again, if $\frac{dx}{dt} = 0$, then $\frac{dx'}{dt} = -\beta v.$ (5)

If $\frac{dx}{dt} = c$, then $\frac{dx'}{dt} = \beta(c-v).$ (6)

Furthermore, if $\frac{dx'}{dt} = c$, it would appear that $c = \beta\left(\frac{dx}{dt} - v\right)$, and therefore $\frac{dx}{dt} = \frac{c+\beta.v}{\beta}$. But this obviously would be impossible as it implies that $\frac{dx}{dt} > c.$ (7)

If $\frac{dx'}{dt} = 0$, then obviously $\frac{dx}{dt} = v.$ (8)

Likewise, if $\frac{dx'}{dt} = \frac{dx}{dt}$, then $\frac{dx}{dt} = \frac{\beta}{(\beta-1)}v.$

In (8) if the body B is at rest relatively to the K' axes, and its velocity is therefore the same as that of the K' axes relatively to the K axes, the time relatively to the two axes is the same.

By simultaneity here is meant of course having the same *now*, that is the same time.

If we consider two such systems of axes K' and K'' moving with velocities v' and v'' relatively to the axes K, then if the coordinates of two bodies B and C relatively to K' and K'' be x'_1 and x'_2, x''_1 and x''_2, and at times t'_1 t'_2 and t''_1 t''_2 respectively, and if x_1, x_2, t_1, and t_2 be the spaces and times relatively to K, then we have

$$t'_1 = \beta\left(t_1 - \frac{v'x}{c^2}\right), \quad t''_1 = \beta'\left(t_1 - \frac{v''x_1}{c^2}\right),$$

$$t'_2 = \beta\left(t_2 - \frac{v'x_2}{c^2}\right), \quad t''_2 = \beta'\left(t_2 - \frac{v''x_2}{c^2}\right).$$

Now suppose that the times at B and C are the same, relatively to K' and K'', then we have $t_1 = t_2$ and $t'_1 = t''_2$.
And we have

$$t'_1 = t''_2 = \beta\left(t_1 - \frac{v'x_1}{c^2}\right) = \beta'\left(t_1 - \frac{v''x_2}{c^2}\right).$$

$$\therefore \qquad (\beta-\beta')t_1 = \frac{1}{c^2}\left(\beta v' x_1 - \beta' v'' x_2\right),$$

or
$$c^2 t_1 = \frac{\beta v' x_1 - \beta' v'' x_2}{(\beta-\beta')}.$$

This is the condition that the three bodies A, B, and C should have the same *now*.

For a single pair, as we have seen,

$$c^2 t = \beta v x/(\beta-1).$$

All bodies satisfying these conditions would have the same *now*.

In so far as all bodies with the exception of the α and β particles from radioactive substances and the electrons in electric conduction, or the motions of these within the atom have very small velocities relatively to the velocity of light, they may therefore be regarded as having approximately the same *now*. The properties further within the atom must be considerably altered, and therefore differ widely from those of gross matter.

It is a question of interest to investigate, on the assumption that time is higgledy-piggledy, the loci of points, having the same time and undergoing the same change in dimensions.

If we represent by Δx the change of length along the x axis, and by Δy that along the y axis, then

$$\Delta x = \frac{c^2}{x} t \Delta t_x, \quad \text{and} \quad \Delta y = \frac{c^2}{y} t \Delta t_y, \text{ where } \Delta t_x = \Delta t_y,$$

so that the velocity v does not enter into the question.

These results can be arrived at quite easily as follows.

We have by the usual notation,

$$x' = \beta(x-vt) \quad \text{and} \quad x = \beta(x'+vt'), \text{ also,}$$

$$t'_x = \beta\left(t - \frac{vx}{c^2}\right) \quad \text{and} \quad t = \beta\left(t'_x + \frac{vx'}{c^2}\right).$$

Hence
$$\Delta t_x = t'_x - t = \beta\left[t - t'_x - \frac{v}{c^2}(x+x')\right],$$

or since
$$x' - x = \Delta x$$

$$\Delta t_x = \frac{-\beta}{(1+\beta)} \frac{v}{c^2}(2x+\Delta x).$$

Similarly,
$$\Delta x = \frac{-\beta}{(1+\beta)} v(2t+\Delta t_x).$$

Therefore $$\frac{\Delta x}{\Delta t_x} = c^2\frac{(2t+\Delta t_x)}{(2x+\Delta x)},$$

a result which is independent of v.

Hence $$(\Delta x)^2+2x\Delta x = c^2[2t\Delta t_x+(\Delta t_x)^2],$$

since $(\Delta x)^2$ is very small and negligible,

$$\Delta x = \frac{c^2t\Delta t_x}{x} \quad \text{or} \quad dx = \frac{c^2tdt_x}{x} = \frac{c^2t}{x}k_1dt,$$

where $$k_1\Delta t_x = \Delta t$$
$$k_2\Delta t_y = \Delta t.$$

Similarly $$dy = c^2tdt_y = \frac{c^2t}{y}k_2dt,$$

and δ = the total contraction

$$\delta^2 = dx^2+dy^2;$$

and since $dx^2+dy^2+dt^2$ is constant

neglecting higher powers of dx and dy than the second,

$$\frac{\delta^2}{c^4t^2dt^2} = \frac{k_1^2}{x^2}+\frac{k_2^2}{y^2}, \quad \text{and if we put} \quad q = \frac{\delta}{c^2\lambda}, \text{ where } \lambda = t\Delta t,$$

we obtain

$$q^2 = \frac{k_1^2}{x^2}+\frac{k_2^2}{y^2}, \quad \text{or} \quad q^2x^2y^2 = k_2^2x^2+k_1^2y^2, \quad \text{or} \quad p^2x^2q^2 = x^2+ky^2$$

where $$p = \frac{q}{k_2} \quad \text{and} \quad k = \frac{k_1}{k_2}.$$

This is a quartic passing through the origin.

Transforming to polar coordinates, we have when $k = 1$

$$p^2\rho^4\cos^2\theta\sin^2\theta = \rho^2$$

or $$\rho^2(p^2\rho^2\cos^2\theta\sin^2\theta-1) = 0.$$

That is, $$\rho = 0 \quad \text{or} \quad p^2\rho^2\cos^2\theta\sin^2\theta-1 = 0.$$

The latter may be written

$$p^2\rho^2\sin^2 2\theta = 4$$

or $$\rho\sin 2\theta = \pm\frac{2}{p}$$

If, therefore, we draw two straight lines parallel to the x-axis at distances $= \pm\frac{2}{p}$, their equations will be

$$\mathrm{P}\sin\Phi = \pm\frac{2}{p}$$

The curve (A) is then the locus of points traced by the bisectors of the angle Φ of length equal to P.

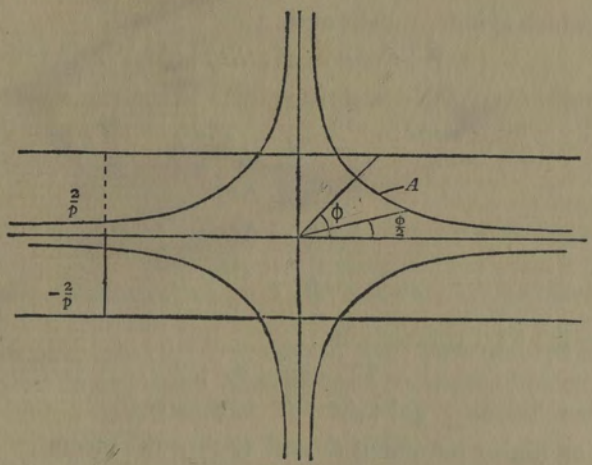


FIG. 3

More generally, if $k \neq 1$,

$$p^2x^2y^2 = x^2+k^2y^2$$

$$x = \rho\cos\theta \quad y = \rho\sin\theta$$

$$p^2\rho^4\cos^2\theta\sin^2\theta = \rho^2[\cos^2\theta+k^2\sin^2\theta]$$

$$\rho^2 = 0$$

$$\rho^2 = \frac{1}{p^2}\left[\frac{1}{\sin^2\theta}+\frac{k^2}{\cos^2\theta}\right]$$

$$\theta = 0 \qquad \rho = \infty$$

$$\theta = \frac{\pi}{2} \qquad \rho = \infty$$

$$\theta = \pi \qquad \rho = \infty$$

$$\theta = \frac{3\pi}{2} \qquad \rho = \infty$$

$$\theta = 2\pi \qquad \rho = \infty.$$

ρ is a minimum when

$$\frac{\cos\theta}{\sin^3\theta}-\frac{k^2\sin\theta}{\cos^3\theta} = 0$$

$$\cos^4\theta = k^2\sin^4\theta$$

$$\tan^4\theta = \frac{1}{k^2}$$

$$\tan\theta = \frac{1}{+\sqrt{\pm k}}$$

If $k = 1$ $\qquad \theta = \frac{\pi}{4}$ i.e. 45°.

Now as p changes, the curves open out, a stage is reached when the quartic breaks up into two simple curves of the second degree.

We have already supposed that when there is an interval of time between different parts of space, as there must be, the locus of points of equal time intervals is a quartic curve whose equation is

$$p^2x^2y^2 = x^2+x^2k^2,$$

the inverse of a conic section, and as the time, or time difference varies, the constant p also changes, so that the family of curves develops continually, until the quartic under certain circumstances breaks up into two lines. If $k^2 = +1$ or $\frac{k_1^2}{k_2^2} = +1$ the curve breaks up into two ellipses, if x and y be inverted as in Chapter XXVI. Here we have a striking resemblance to the phenomenon of karyokinesis in Cytology. With the change in the time difference branches of the characteristic curves spread out or contract, as the case might be, and in the latter case the quartic gradually breaks up into two ellipses. When the two constituent portions separate so that each goes its own way in actual reality, then each is at liberty to combine with another such globule, should the circumstances be favourable.

More rigorously we may proceed as follows:

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right) \qquad t_2'' = \beta\left(t_2 - \frac{vx_2}{c^2}\right)$$

$$x_1' = \beta(x_1 - vt_1) \qquad x_2'' = \beta(x_2 - vt_2)$$

(*a*) If $t_1 = t_2$

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right) \qquad t_2'' = \beta\left(t_1 - \frac{vx_2}{c^2}\right)$$

$$x_1' = \beta(x_1 - vt_1) \qquad x_2'' = \beta(x_2 - vt_1)$$

$$t_1' - t_2'' = -\frac{\beta v}{c^2}(x_1 - x_2) = -\frac{v}{c^2}(x_1' - x_2'')$$

$$x_1' - x_2'' = \beta(x_1 - x_2).$$

(*b*) If $t_1' = t_2''$

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right) \qquad t_1' = \beta\left(t_2 - \frac{vx_2}{c^2}\right)$$

$$x_1' = \beta(x_1 - vt_1) \qquad x_2'' = \beta(x_2 - vt_2)$$

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right) = \beta\left(t_2 - \frac{vx_2}{c^2}\right)$$

$$t_1 - t_2 = \frac{v}{c^2}(x_1 - x_2)$$

$$x_1' - x_2'' = \beta(x_1 - x_2) - v\beta(t_1 - t_2)$$

$$= \beta(x_1 - x_2) - \frac{\beta v^2}{c^2}(x_1 - x_2)$$

$$= \beta\left(1 - \frac{v^2}{c^2}\right)(x_1 - x_2)$$

$$= \frac{1}{\beta}(x_1 - x_2).$$

(*c*)

$$t' = \beta\left(t - \frac{v}{c^2}x\right) \qquad t = \beta\left(t' + \frac{v}{c^2}x'\right)$$

$$x' = \beta(x - vt) \qquad x = \beta(x' + vt')$$

$$t' = t + \delta t$$

$$t + \delta t = \beta\left(t - \frac{v}{c^2}x\right)$$

$$t = \beta\left[t + \delta t + \frac{v}{c^2}(x + \delta x)\right]$$

$\therefore$

$$\delta t = -\beta\delta t - 2\beta\frac{v}{c^2}x - \beta\frac{v}{c^2}\delta x$$

$$(1+\beta)\delta t = -\beta\frac{v}{c^2}(2x + \delta x)$$

$$\delta t = \frac{-\beta v}{c^2(1+\beta)}(2x + \delta x)$$

$$x + \delta x = \beta(x - vt) \qquad x = \beta[x + \delta x + v(t + \delta t)]$$

$$\delta x = -\beta\delta x - v\beta[2t + \delta t]$$

$$\delta x = -\frac{v\beta}{(1+\beta)}[2t + \delta t]$$

$$\therefore \qquad \frac{\delta x}{\delta t} = \frac{c^2(2t+\delta t)}{2x+\delta x}$$

$$\therefore \qquad 2x\delta x+\delta x^2 = c^2[2t\delta t+\delta t^2]$$

$$x\delta x = c^2 t\delta t \qquad \delta x = \frac{c^2}{x}t\delta t$$

$$\delta t_x = k_1\delta t \qquad \delta x = \frac{c^2}{x}t\delta t_x = \frac{k_1c^2}{x}t\delta t$$

$$\delta t_y = \mathrm{k}_2\delta t \qquad \delta y = \frac{c^2}{y}t\delta t_y = \frac{k_2c^2}{y}t\delta t$$

$$\delta^2 = \delta x^2+\delta y^2+\delta t^2 = c^4t^2\delta t^2\left[\frac{k_1^2}{x^2}+\frac{k_2^2}{y^2}+\frac{1}{c^4t^2}\right] = \text{const.}$$

$$\frac{1}{x^2}+\frac{p^2}{y^2} = \text{const.}$$

$$qx^2y^2 = p^2x^2+y^2.$$

Supposing that $\delta t_x = f_1(\delta t) \qquad \delta t_y = f_2(\delta t)$

then $$\delta x = \frac{c^2}{x}tf_1(\delta t) \quad \delta_y = \frac{c^2}{y}tf_2(\delta t)$$

$$\delta s^2 = \delta x^2+\delta y^2+\delta t^2$$

$$= c^4t^2\left[\frac{\{f_1(\delta t)\}^2}{x^2}+\frac{\{f_2(\delta t)\}^2}{y^2}+\frac{\delta t^2}{c^4t^2}\right] = 0$$

$$(x+\delta x)^2 = x^2+c^2(2t\delta t+\delta t^2)$$

$$2x\delta x+\delta x^2 = 2c^2t\delta t+c^2\delta t^2$$

$$2y\delta y+\delta y^2 = 2c^2t\delta t+c^2\delta t^2$$

$$\delta x^2+\delta y^2-c^2\delta t^2 = 0$$

$$2x\delta x+2y\delta y = 2c^2t\delta t_x+2c^2t\delta t_y+c^2\delta t_x^2+c^2\delta t_y^2$$

$$\rho^2 = x^2+y^2 = 2c^2\int[(t\delta t_x+\delta t_x^2)-(t\delta t_y+\delta t_y^2)]$$

δt_x and δt_y, may be considered functions of $\cos\theta$ and $\sin\theta$ as in Fourier's Series. Under such circumstances the figure traced out by ρ will be a complicated one, but determined.

If $\quad x' = \beta(x-vt), \quad$ and $\quad t' = \beta\left(t-\frac{vx}{c^2}\right)$, then

$$\Delta t = \tau = t'-t = (\beta-1).t-\beta\cdot\frac{vx}{c^2} = \beta\cdot\left(\frac{(\beta-1)}{\beta}t-\frac{vx}{c^2}\right).$$

Putting

$$t'' = t - \frac{t}{\beta} = \left(\frac{\beta - 1}{\beta}\right).t, \quad \text{i.e.} \quad \left(\frac{t - t''}{t}\right) = \frac{1}{\beta}, \quad \text{or} \quad Dt = -\frac{t}{\beta},$$

we get $$\Delta t = \tau = \beta\left(t'' - \frac{vx}{c^2}\right).$$

Similarly $$\Delta x = \beta(x'' - vt), \quad \text{where} \quad x'' = x - \frac{x}{\beta}.$$

Hence if the time at the origin is altered by $\frac{t}{\beta}$ the time at the point originally t' becomes $(t' - t)$ or Δt, and conversely t'' the new time according to the moving axes becomes

$$t'' = \beta\left(t - t' + \frac{vx}{c^2}\right).$$

Moreover,

$$\Delta t'' = t' - t'' = t' - \beta\left(t - t' + \frac{vx}{c_2^2}\right)(1+\beta)t' - \beta\left(t + \frac{vx}{c^2}\right)$$

$$= (1+\beta).t' - t' = \beta t'.$$

$$\therefore \qquad \Delta t'' = \beta.t'.$$

When only the time is altered at the point O, by $\frac{t}{\beta}$,

$$x'' = \beta\left(x - vt + \frac{vt}{\beta}\right) = \beta(x - vt) + vt = x' + vt, \quad \text{or} \quad x'' - x' = vt.$$

Hence if the time at the origin O is diminished by $\frac{t}{\beta}$ the contraction with reference to the moving axes is equal to the distance of O', the moving origin, from the origin O.

Again, if $t' = \beta\left(t - \frac{vx}{c^2}\right)$, and $t'' = \beta\left(t_1 - \frac{vx}{c^2}\right)$, where t_1 is the altered time at the fixed origin, then $t' - t'' = \beta(t - t_1)$, and likewise

$$x' - x'' = -\beta v(t - t_1), \quad \text{and} \quad \frac{x' - x''}{t' - t''} = v \quad \text{or} \quad \frac{\delta x'}{\delta t''} = -v.$$

A retardation of time at O is followed by a contraction at x', but the effect could obviously be manifested when the change of time at O shall have been communicated to O' with the velocity of light.

Every time that there is a reversal of motion by the stoppage and rebound of an electron a corresponding change of time takes place. The impact of an electron or proton with an atom results in a time disturbance. Change of time means change of place, i.e.

motion, since space is the order of coexistence of monads forming an event (see p. 286) then the space-time line is altered and the position of the monad whose time is altered must needs be likewise changed relatively to other monads. That a wave should be propagated is 'a brutal fact'. It constitutes a rearrangement of all monads in space, and it is propagated at a definite velocity, i.e. the velocity of light, about 300,000 kilometres per second.

CHAPTER XXV

RELATIVITY AND LIFE

THE corpuscular theory of light according to which the *quanta* play a part similar to that of electrons and molecules, though on a far smaller scale, opens up vistas almost too wonderful to contemplate. For the phenomena of the interference of light present an appearance so strikingly similar to those of karyokinesis of cells, suggesting that the latter are due to periodic disturbances of a like nature as Leduc has succeeded in demonstrating with his artificial cells, by the periodic deposition of precipitates. I have long maintained, however, that if the nuclei radiate particles, whether quanta as in the case of light, or electrons as in radioactive matter, or diffuse molecularly as in the case of osmosis, &c., interference effects may be produced in a rhythmical manner, the actual velocity and wave-length depending upon the velocity of propagation period of the vibration and properties of the medium.

The beautiful effect produced by the interference of light waves, and by the polarization of light, has its analogue in the superposition and interference of certain rhythmical actions, periodic wave disturbances if we prefer to call them so, in the living cell. Stephane Leduc in his *Mechanism of Life* gives some interesting illustrations which exhibit the effects of a rhythmical deposition. The movements though slow and of course of an entirely different order of magnitude from those of light, yet can be made to produce interference effects similar to those of light, by diffraction and polarization. The process is then largely mechanical, though not altogether so. Biologists have vacillated between the purely mechanical and purely vitalistic conceptions of the process. The two theories, however, appear to be like the Hegelian antinomies, reconcilable. Neither is, strictly speaking, true. The truth lies in a higher synthesis formed from the combination of the two apparently contradictory hypotheses. In the new ideas of time and space I have endeavoured to put forward, Vitalism is shown to be a consequence of the evolutionary process, and Mechanism is likewise the means of effecting the development of cellular life. If the nucleus radiates it can give rise, as we have seen, to such interference effects. But also to far more, for the breaking up of the nuclear vibrations of time differences into their Fourier components enables the non-spatial monads to respond as if from nowhere in a particular place. The aggregate of apparently superposed monads represents a creation which, whether sponta-

neously created or not, can manifest the characteristics of living matter.

The striking analogy afforded through the polarization of light by crystals and the phenomenon of karyokinesis in living cells is

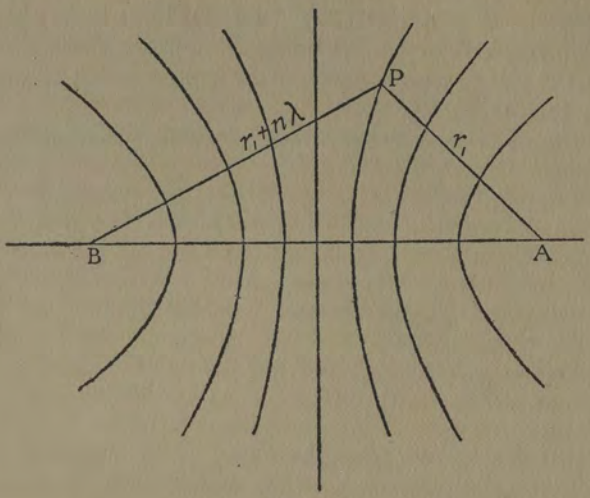


FIG. 4

noteworthy, and merits greater attention than has hitherto been given to it. It suggests to anybody with a vivid imagination the possible relationship through some common physical process in the two effects.

If A and B be the two foci of an hyperbola as in Fig. 4 and P be any point on one of these, and the radii vectors, $AP = r_1$, $BP = r_2$, then $r_2 - r_1 = \text{Constant} = n\lambda$, let us suppose, where λ is the common wave-length of the radiation from the two points A and B, supposed to be very close together. These radiations may be of any nature whatsoever, from light rays to waves of colloid diffusion, causing precipitation in interference, as in Leduc's experiments. The polar equation $r_2 - r_1 = n\lambda$ may be transformed to Cartesian coordinates and written $xy = k^2$ with the asymptotes as axes.

Now for a rectangular hyperbola $\frac{AB}{\sqrt{2}} = n\lambda$ of the order of wave-length we may write

$$x = \frac{r}{2}(e^{i\theta} + e^{-i\theta}), \qquad y = \frac{r}{2i}(e^{i\theta} - e^{-i\theta}),$$

where $i^2 = -1$.

Hence, $(e^{2i\theta} - e^{-2i\theta}) = -\frac{4k^2}{r^2}$, and putting $X = e^{2i\theta}$, $Y = e^{-2i\theta}$, we obtain

$$X - Y = -\frac{4k^2}{r^2}, \quad \text{if} \quad P = \frac{Xr^2}{4k^2}, \quad Q = \frac{Yr^2}{4k^2}, \quad Q - P = 1.$$

So that P and Q when whole numbers must be consecutive, and one odd, the other even.

If, therefore, P and Q are units following Mendel's law as in p. 355, since

$$XY = 1, \quad \text{and} \quad X^2 + \frac{4k^2}{r^2}X - 1 = 0, \quad \text{and} \quad Y^2 - \frac{4k^2}{r^2}Y - 1 = 0,$$

$$P = \frac{Xr^2}{4k^2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{r^4}{4k^4}},$$

$$Q = \frac{Yr^2}{4k^2} = +\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{r^4}{4k^4}},$$

we get that $$(2P+1)^2 - 1 = \frac{4^4}{rk^4},$$

and $$(2Q-1)^2 - 1 = \frac{r^4}{4k^4}.$$

Consequently for every value of P or Q, there are corresponding values of $\frac{r}{k}$.

There are thus a family of hyperbolae traced by giving P and therefore Q a series of values $(1, 2, 3, \ldots, n)$.

If P and say P' belong to different families we obtain

$$Y_1 = P - P' = \frac{Xr^2}{4}\left(\frac{1}{k^2} - \frac{1}{k'^2}\right),$$

a law equivalent to Rydberg's already found to hold for similar circumstances when a displacement of an electron takes place within the atom. Here, however, we are dealing with something far more general. The ψ and ψ functions in wave-mechanics can be made to take the place of the exponential functions here employed.

The point is that from the units P and Q an effect can be produced 'as if' there was a radiation from the points A and B. But this is merely the result of the transformation of coordinates,

and like the effects produced in images. Physical phenomena in the hypothetical space can therefore be always supposed to be the outcome of correlative occurrences in hyper non-spatial modes of being, in which monads interact in time without any spatial interaction whatsoever. This view gives to the monadology an entirely new aspect, and affords a clue which I trust will enable us to apprehend the possibility of phenomena as yet inexplicable.

The function of the centrosome in the development of the living cell has never been understood. It appears to be an essential and most important part of the mechanism of the cell. But no one has so far as I am aware ever offered an explanation of its remarkable behaviour, at any rate on purely physical grounds. The influence that it exerts on the nucleus in bringing about subdivision is an enigma which biologists have very rightly regarded as beyond their ken. The grouping of chromosomes, on the other hand, though less mysterious owing to their more symmetrical and rhythmical behaviour, may be expected to depend upon the distribution of time in the various constituents relatively to the nucleus itself and the rest of the cell. The arrangements then should assume extremely complicated relationships, though none the less amenable to definite laws. If we consider the relative retardations in the various parts of the cell, both as regards dimensions and time, we may arrive at relationships between the chromosomes, nuclei, and inert matter. The centrosome, however, apart from these must be regarded as exerting by its mere presence a remarkable action.

The centrosome appears to play a part somewhat analogous to that of enzymes in chemical processes, although obviously different yet suggestive of actual directiveness in some respects. The monad represented by the centrosome appears to be endowed with some peculiar power which, in the light of the considerations given above, may result from a selection of partials in the Fourier series belonging to the organism. The *before* and *after* qualities of the time-acceleration affect the relative selective powers of every particular monad. And since in a space-time continuum every point has its own space-time, then any two monads differing considerably in time-acceleration though approximately of the same time and place would tend to separate, and those having time-accelerations in different times and places would conversely tend to occupy the same time and place.

Two monads, therefore, centrosome, nuclei, or chromosomes as the case might be, would behave towards each other in the following manner.

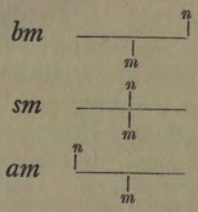
(1) Having the same time but different time-accelerations. (Repelled.)

(2) Having the same time, and same time-acceleration. (No effect.)

(3) Different times, and same time-accelerations. (Attracted.)

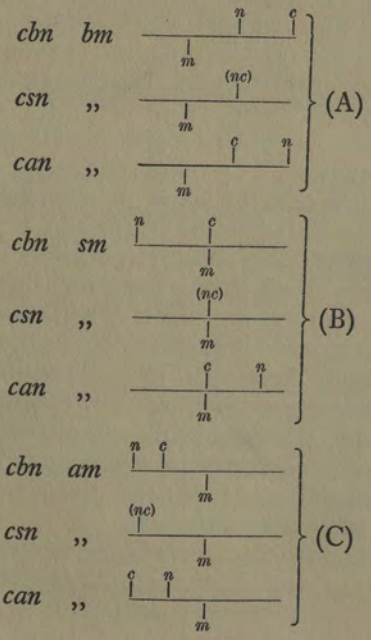
(4) Different times, but different time-accelerations. (Attracted or repelled, as the case might be.)

If we regard n = nucleus, c = chromosome, m = inert matter, the relative attitudes of each towards each may be represented as follows, taking in the first instance the relations of the nucleus and inert matter alone.



Where $b.m$, stands for before inert matter, $s.m$, for simultaneous with inert matter, and $a.m$, after inert matter.

And the relations of the nucleus with any particular chromosome,



If we neglect the case B.2, i.e. $c.s.n : s.m$, we are left with eight possible cases:

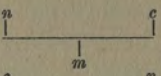
$$m-n-c,\quad m-(nc),\quad m-c-n,\quad n-(mc),\quad (mc)-n,$$
$$n-c-m,\quad (nc)-m,\quad c-n-m.$$

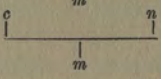
And if the four cases of simultaneity amongst these be again eliminated, we are left with the four,

$$m-n-c,\quad m-c-n,\quad n-c-m,\quad c-n-m.$$

With these elements which we shall call D_1, D_2, D_3, D_4, respectively, two other cases have to be considered, namely,

$$n-m-c,\quad c-m-n$$

$c.b.n\ b.a.m$ 

$c.a.n\ b.a.m$ 

(D)

These two, however, which we may call D_5, D_6, form a class apart. Here we may have a flow of matter between the nucleus and chromosome in two opposite manners or directions, the two nuclei behaving like positive and negative units.

We have thus presented to us a series of problems in the Theory of Groups. The fact that the order now enters into the consideration of the combinations reduces the problems to one requiring a particular kind of algebra as in Vector Analysis, in the products of axial vectors.

CHAPTER XXVI

HEREDITY AND VARIATION

IF two kinds of sensations vary in such a manner that they can be plotted graphically and represented by a straight line, then the stimuli are represented graphically by a hyperbola.

The transformation of coordinates may be of any nature whatsoever. Thus we may put $x = f_1(X)$ and $y = f_2(Y)$; the curve

$$F(x,y) = 0 \quad \text{becomes} \quad F\{f_1(X),\ f_2(Y)\} = 0.$$

Hence if x and y undergo a change then X and Y will likewise do so; but the changes are of an entirely different kind. The logarithmic transformations, bearing upon the psychological phenomena we have been considering, constitute only a particular class of transformations.

It would be possible to transform all the detailed structures and convolutions into any figure, *if we but knew the functions*, $F, f_1 f_2$, which unfortunately we do not know.

It is not merely space that becomes curved, as in non-Euclidean geometry, but time or duration if altered would give rise to results that, however complicated at first sight, may be but the results of quite simple laws in another system of coordinates from which they have been transformed.

It is obvious that through a system of inversion in which the coordinates are replaced, for instance, by their logarithms, the inverted curve will be such that every point on it will have a corresponding point on the curve of which it is the inverse. Yet the two curves will resemble each other no more than an embryo resembles the developed organism. But the corresponding points in the two systems will have some striking characteristics.

As we have seen, a hyperbola can be converted into a straight line, and vice versa, by a process of logarithmic transformation, or else inversion. So that the peculiar paths we witness in the phenomenon of karyokinesis of a living cell (see Fig. 332) may be regarded as equivalent to a purely rectilineal motion in another system of coordinates. Likewise, the lines of magnetic force between the two poles of a magnet, or indeed the Faraday lines of force between two electrified bodies. Such curves can be inverted into simple ones, by a transformation of coordinates as in the Theory of Groups.

A transformation from a world of material objects to a world of

perceptions would thus establish a connexion and correlation between phenomena and laws in two apparently different spheres of Science such as Psychology and Physics on the one hand, and Biology, or Cytology, and Physics on the other.

This conception of space and time affords an insight into the inner nature of the monad, and enables many phenomena to be accounted for that would otherwise remain absolutely unrelated to physical effects, as belonging to distinct and water-tight compartments of science.

If the change in dimension is continuous and not perfectly uniform for all kinds of matter, then that which lagged behind, so to speak, would experience a force, outwards from itself, a 'vital force' as it were, tending to grow or enlarge itself relatively to other bodies. That is to say, it would manifest what Bergson calls an *élan vital*, which he has not accounted for in living as distinct from ordinary matter.

Similarly as regards the concept of time. If everything is getting on the whole slower and slower, or faster and faster, and some things are retarded or advanced over others, they will obviously appear to be spontaneously active. Hence, if there is a retardation in the diminution of size in an object and at the same time a retardation in its change of time, there would be an active tendency to grow and to move spontaneously.*

Anything of the nature of a mental process may tend to retard contraction, and thus bring about the effect of an apparent force tending to develop in certain directions rather than in others, owing to the particular shape or form of the matter so retarded.

A lower world tends to assert itself in a higher. Living matter 'reflects', so to speak, the forms of an unseen world from which it is drawn out, by the retarding influence of the action of mind or will upon the time, on the one hand, and upon the idea of dimension on the other, giving rise to two distinct kinds of 'force', physico-psychological and physiological.

The exponential functions can always be expressed in sines or cosines thus

$$e^{\theta} = \cos i\theta - i \sin i\theta, \quad e^{-\theta} = \cos i\theta + i \sin i\theta,$$

and these are resolvable into Fourier components.

* If the Earth, for instance, contracts according to this conception more rapidly than the living bodies on it, then these would experience what is known as a *geotropic* force in a direction opposite to that of gravity. For instance all plants and trees grow vertically, and not perpendicularly to the surface of a mountain. This phenomen of *geotropism* has never been properly explained.

We assume absolute uniformity where there is none and heterogeneity accounts for the observed facts if there is only an approximate uniformity.

If therefore the time-function of a monad, that is, its manner of acceleration or retardation relatively to other monads, is determined, it can be resolved into its Fourier partials, to which other monads will respond, forming an organism in 'space'. The growth and subsequent development of the organism will depend upon the 'time' of the fundamental monad. Its figure or shape in space will be but a transformed relationship of classes, much as Peano's geometry from the primitive laws of arithmetic. But here we have a physical resonance like that of so many tuning-forks or electric valves to a particular tune. Our world is a majestic concert produced by such tuning-forks or valves, and life only the manifestation of the more excited monads amongst them. Each such monad has its own peculiar 'tune' resolvable in time-vibration, like the spectrum of an element the atoms of which may be regarded as monads. If light be the group 'time-wave' we have supposed it to be, the laws of one would be analogous to those of the other, though the differences in the organism involve a time of a far more complicated kind.

The coordinates of any curve may be resolved as we know into the sum of a series of sines and cosines of a 'fundamental' x, and its 'harmonics' nx. This is known as Fourier's Series. According to this theorem any vibration in general may be broken up into a fundamental and its harmonics. Thus any periodic curve can be broken up into a primary sine curve of variable x and a series of such sine curves for nx, where $n = 1; 2; 3; 4 \ldots$.
Thus

$$F(x) = a_0 + a_1 \sin x + a_2 \sin 2x + \ldots a_{n-1} \sin (n-1)x + \\ + b_1 \cos x + b_2 \cos 2x + \ldots b_n \cos nx.$$

$F(x)$ thus recurs periodically, but if we confine ourselves to a single oscillation of this curve, or any chosen portion of it, this can be regarded as made up of harmonics so far as they can fit into the portion of the curve under consideration; but the fundamental and lower harmonics may be represented in part. That is the whole disturbance—for we may regard the curve as representing one—the total disturbance may be aperiodic, but some of the partials at least, that is, the ones of shorter wave-length, would be periodic, and the shorter the wave-length the greater the number of oscillations executed in the time of the disturbance.

So long as the curve representing the disturbance has a definite shape, it is resolvable into partials by Fourier's Series, although the disturbance is not repeated periodically. The lower partials may be, or may not be, periodic, but the higher ones are clearly so.

Any non-periodic disturbance may be regarded as a portion of a fictitious periodic one which if it actually took place would, no doubt, be resolvable into a fundamental and its harmonic partials. Hence during the non-periodic disturbance, these partials will operate, but only during that interval of time of the actual disturbance, since if they were continued the total disturbance would become periodic. (See 'Methods and Means of controlling the Operation of Machines by Vocal Sounds and the Automatic Printing of Telephone Messages', by the author, *Patent Specification* No. 244, 138.)

Now since

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \text{and} \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i},$$

it follows that

$$F(x) = A_0+A_1e^{ix}+A_2e^{2ix}+ \dots A_{n-1}e^{(n-1)ix}+ \\ +B_1e^{-ix}+B_2e^{-2ix}+ \dots B_ne^{-ixn}.$$

In the psychological case we have been considering

$$e^P = Kv \quad \text{or} \quad K'\sigma,$$

where P is the sensation, and v = the stimulus, σ = the duration. Thus the sensation can be represented as a hyperbolic function.

Similarly, as we have seen, any curve can be transformed by a similar process into another corresponding one in an entirely different system of coordinates.

Furthermore the variations in dimension, and those of time we have been considering, or the rate of variations of such for each atom or molecule, may be periodic, and the resultant effect on an aggregate of these would be approximately represented by Fourier's Series as a periodic variation; the periods being fractions of a second or years as the case may be, according to the conditions of the problem. But the resultant vibration, and the curve representing it, may be extremely complicated.

Any force, whether vital or otherwise, and any sensation however complex, thus admits of resolution into constituent partials; and any curve may not only be resolved into a fundamental and its harmonics, but by transformation of coordinates it may be seen to be deduced from a curve of an entirely different nature.

The replica of some forms in a world of smaller dimensions than our own though differing in the two scales as much as a translation in one language differs from the original in another, or the written from the spoken word, is as though thought could wed

itself with speech or one world with the other. We may also express $x_0x_1x_2x_3x_4 \ldots x_n$, in the form of a series thus

$$= x_0+x_0(x_1-1)+x_0x_1(x_2-1)+x_0x_1x_2.(x_3-1)+ \\ + \ldots x_0x_1x_2x_3 \ldots x_{n-1}(x_n-1).$$

Any function can be expanded thus

$$F(x) = \frac{1}{\frac{1}{F(x)}} = \frac{1}{\phi(x)},$$

where

$$\phi(x) = \frac{1}{F(x)} = \{F(x)\}^{-1}.$$

$$F(x) = \frac{1}{\phi(x)} = \frac{1}{\phi(x)}\frac{\phi(x)+\psi(x)}{\phi(x)+\psi(x)} = \frac{1}{\phi(x)+\psi(x)} + \frac{\psi(x)}{\phi(x)+\psi(x)} - \frac{1}{\phi x}$$

$$= \frac{1}{\phi(x)+\psi(x)} + \frac{\psi(x)[\phi(x)+\psi(x)]}{\phi(x)[\phi(x)+\psi(x)]^2}$$

$$= \frac{1}{\phi(x)+\psi(x)} + \frac{\psi(x)}{[\phi(x)+\psi(x)]^2} + \frac{[\psi(x)]^2[\phi(x)+\psi(x)]}{\phi(x)[\phi(x)+\psi(x)]^3}$$

$$= \frac{1}{\phi(x)+\psi(x)} + \frac{\psi(x)}{[\phi(x)+\psi(x)]^2} + \frac{[\psi(x)]^2}{[\phi(x)+\psi(x)]^3} + \\ + \frac{[\psi(x)]^3}{\phi x[\phi(x)+\psi(x)]^3} + \ldots.$$

Therefore

$$F(x) = \frac{1}{\phi(x)} = \frac{1}{[\phi(x)+\psi(x)]} + \frac{\psi(x)}{[\phi(x)+\psi(x)]^2} + \frac{[\psi(x)]^2}{[\phi(x)+\psi(x)]^3} + \ldots$$

If

$$\psi(x) = 1,$$

$$F(x) = \frac{1}{\phi(x)} = \frac{1}{[\phi(x)+1]} + \frac{1}{[\phi(x)+1]^2} + \frac{1}{[\phi(x)+1]^3} + \ldots,$$

when

$$F(x)\phi(x) = 1.$$

If

$$\psi\bar{\psi} = 1,$$

therefore

$$\psi = \frac{1}{\bar{\psi}} = \frac{1}{\bar{\psi}+1} + \frac{1}{(\bar{\psi}+1)^2} + \frac{1}{(\bar{\psi}+1)^3} + \ldots$$

$$\bar{\psi} = \frac{1}{\psi} = \frac{1}{\psi+1} + \frac{1}{(\psi+1)^2} + \frac{1}{(\psi+1)^3} + \ldots$$

$$= \frac{1}{\psi+x} + \frac{x}{(\psi+x)^2} + \frac{x^2}{(\psi+x)^3} + \ldots \text{ etc.}$$

The idea here is that if there is a continuous change of dimension downwards, a change in which the whole creation almost synchronously takes part, there is nevertheless between individual monads a normal average: and some do in fact in the long run lag behind. Hence one Universe, the material Universe, does pass through, as it were, another changing less rapidly, so that as it passes through the various scales there is an emergence or repetition of approximately the same events in living matter as incorporated by the phenomenon of reproduction. Yet the event only takes place when there is a variation from the normal process, for the mark of the rhythmic process then only acquires its assertive power.

Let it not be forgotten that we have been considering the possibility of a change of dimensions, such that the World in which we live might be altered to, say, n times or to $\frac{1}{n}$th, its present size, without our being able to acquire any knowledge whatsoever of the change.

Now it is a well-known fact* that the dimensions of the stellar universe are $10^{22} = c^2$ (the square of the velocity of light) that of the world in which we live (at any rate as it was then known to be, for the structure of many spiral nebulae was not then considered to be of the nature of Milky Ways in themselves), and this 10^{22} that of the atomic world, so that if the Earth were reduced to the size of an atom, then the Milky Way would be reduced to the size of a bishop's ring. The three worlds are in geometrical progression $10^{44} : 10^{22} : 1$, where 1 is the diameter of an electron.

Thus the radius of an electron $= 10^{-14}$ cm.

The radius of the Earth

$$= 4000 \times 5280 \times 30 \text{ cm.} = 4 \times 5{\cdot}28 \times 3 \times 10^7 = 6{\cdot}3 \times 10^8.$$

$$\frac{\text{Radius of an electron}}{\text{Radius of the Earth}} = \frac{10^{-14}}{6{\cdot}3 \times 10^8} = \frac{1}{6{\cdot}3} \times 10^{-22}.$$

The velocity of light $= 3 \times 10^{10}$ per second. Hence the number of cm. in a light year

$$= 3 \times 10^{10} \times 365 \times 24 \times 60 \times 60 = 9.4243 \times 10^{17}.$$

Now according to the most recent measurements, the distance of the farthest stars in the Milky Way is about 30,000 light-years $= 3 \times 10^4$. That is, the distance of the farthest stars in the Milky Way is of the order $= 2{\cdot}82 \times 10^{22}$, so that if the Milky Way were

* I believe that it was first emphasized by E. E. Fournier d'Albe, in his *Two New Worlds* (1909).

reduced in the proportion 10^{-22}, it would be no more than 2·8 cm. in diameter, or $1\frac{1}{8}''$. Hence

(the dimension of the Milky Way)
: (the dimension of a bishop's ring)
:: (the dimension of the Earth)
:(the dimension of an electron).

There are 10^{18} cm. in a light-year, and the distance of the remotest stars in the Milky Way is 3×10^4 light-years, then as we have seen the distance of the remotest stars in our Milky Way is of the order $= 3 \times 10^{22}$ cm.

But according to the latest views of the structure of spiral nebulae these are in their turn Milky Ways, and if they are as far from our Milky Way as the remotest star of our own system is from us—a very low limit to impose—the distances between these Milky Ways or universes would be of the order

$$3 \times 10^4 = 9 \times 10^{26} = 10^{27}.$$

Curiously enough the wave-length of a time-change wave is of the order 10^{-25} cm.*

Further: The Universe of Milky Ways $= 10^{27}$ cm.
The Solar System $= 10^{14}$ cm.
An ordinary Earthy object $= 10^0$ cm. or 1
An electron $= 10^{-14}$ cm.
A photon or chronion $= 10^{-27}$ cm.†

The ratios of these are of the order 10^{13} or 10^{14} approximately. Again the order of magnitude of the solar system being 10^{14} and that of an atom 10^{-8} cm., it follows that the atomic to the solar systems are in the proportion

$$10^{-22} : 1.$$

Whether we take the change of scale to be from 10^{22} or 10^{13}, though the latter seems preferable (and it should be borne in mind that the limits may be very wide in this respect; for instance, $10^{\pm 3}$ would not appear extravagant, so that a scale of 10^{11} would seem possible), the relationship is important and striking. We shall take $10^{13} : 1$.

* It is easily found that the difference of time between an electron and the proton in an atom radiating waves of light is of the order 10^{-35} cm. so that the wave-length of the 'time-difference wave' is 10^{-25} cm. Light would be a group of such 'time-waves'. (See Chapter XXIV, sec. 8.)

† I cannot say whether the photon and what I here call a chronion can be identified; but it seems likely that the latter is still smaller. We may at present assume it to be the photon.

For example any number can be broken up into a series of ascending powers *ad infinitum*, we may write then

$$10^{13} = \frac{10^{13}}{10^{13}+1} + \left(\frac{10^{13}}{10^{13}+1}\right)^2 + \left(\frac{10^{13}}{10^{13}+1}\right)^3 + \dots .$$

Similarly,

$$\frac{1}{10^{13}} = \frac{1}{(10^{13}+1)} + \frac{1}{(10^{13}+1)^2} + \frac{1}{(10^{13}+1)^3} + \dots$$

Note the contrast between 10^{13} and 1.

The phenomena of growth, reproduction, and heredity may be compared to such recurring series. For the numbers that repeat themselves may in fact be very far apart, as in the repeating decimal

$$0\cdot\dot{4}69821459386469821459386469821459386\dots .$$

There may be a thousand, a million, a billion, or a trillion terms in a recurring group, but that would not prevent it from recurring for ever.

Hence if, for example, $\frac{n}{99999\dots99}$ be a recurring series, where n is a very large number, and if $\frac{n'}{99999\dots99}$ be any other such series, then $\frac{n+n'}{99999\dots99}$ represents a recurring series in which the recurring figures are the sum of the constituents in the previous recurring cases.

Thus if two such groups of recurring terms in an infinite series of scales or dimensions be combined in such a manner that the terms are added in one scale, they will be added in all scales for ever. Further slight or considerable variations can arise from the combination of two such series.

Now according to this if two groups be combined in a manner analogous to the process of multiplication such that every constituent of one group combines with every constituent of the other, then the logarithm of the combination forms a recurring and converging series.

Thus

$$\frac{m}{10^{27}} = m\left(\frac{1}{10^{27}+1} + \frac{1}{(10^{27}+1)^2} + \dots\right).$$

$$\frac{m'}{10^{27}} = m'\left(\frac{1}{10^{27}+1} + \frac{1}{(10^{27}+1)^2} + \dots\right).$$

Hence if $m = \log p$, and $m' = \log q$,

$$\log pq^{\frac{1}{n}} = \frac{\log p + \log q}{n} = (m+m')\left(\frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \ldots\right)$$

$$\log pq^{10^{-22}} = (m+n)\left(\frac{1}{(10^{22}+1)} + \frac{1}{(10^{22}+1)^2} + \ldots\right).$$

Thus the union of groups, in the aforesaid manner, whether they be gametes of cells or molecular groups of any kind, that combine in accordance with the associative law in multiplication, i.e. that

$$(a+b)(c+d) = (ab+ad+bc+bd),$$

if transformed so that $m = \log p$ or $e^m = \mathrm{p}$, and $e^{m'} = q$, etc., forms a series that for ever repeats itself, however simple or complicated the groups might be.

The actual dimension need not frighten us.

Let m represent a group such, for instance, as might correspond to the grouping that makes up a plant, let us say an oak. Then if m' be another such, differing slightly from m; $m+m'$ will contain a recurring series differing slightly from either but resembling both. Clearly when the scale is not 10, but $(10^{22}+1)$ the possibilities become very great indeed.

So long as the grouping takes place in a definite order whatever the number of the constituents, provided they are finite, the recurrence can be expressed in a series of powers of ascending or descending orders.

It would appear from the above considerations that when two groups combine, slight variations result in the combinations, but that under certain circumstances the variations may be considerable, i.e. equivalent to mutations, in the case of a cell.

The application of the theory of numbers to the combinations of biological determinants by the peculiar properties of *primes* as grouped and worked out in the manner I have shown is even more noteworthy. Now the logic of classes presents similar anomalies. But we shall confine ourselves for the present to the theory of numbers as ordered classes.

CHAPTER XXVII

VARIATION BY MUTATION

IF we designate all odd numbers by S and all even numbers by T, then it is easy to see that each class can be subdivided into three others the relations between which are extremely remarkable.

Thus all odd numbers S may be regarded as belonging to one or other of the three series S_1, S_2, S_3, where

$$S_1 = 1+6l$$
$$S_2 = 5+6m$$
$$S_3 = 3+6n.$$

S_1 and S_2 are *primes* (except when they happen to be divisible by 5 or 9.

S_3 are all odd numbers and at the same time *not* primes, since every member of the series contains the factor 3. These are called *composite* odd numbers.

We have $S_1 = 1+6l$, 1, 7, 13, 19, *25*, 31, 37, 43, 49, *55*, 61, 67, 73, 79, *85*, *91*, 97, 103, &c.,

and $S_2 = 5+6m$ = 5, 11, 17, 23, 29, *35*, 41, 47, 53, 59, *65*, 71, *77*, 83, 89, *95*, 101, 107, &c.

Odd numbers which are not primes but composites, apart from those in italics in the series above

$S_3 = 3+6m$ 3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99, 105, 111, &c.

We may likewise regard all even numbers as belonging to one or other of the three series T_1, T_2, T_3, where

$$T_1 = S_1+1 = 2+6l,$$
$$T_2 = S_2+1 = 6+6m,$$
$$T_3 = S_3+1 = 4+6n.$$

It is easy to show that (1) $S_1S_1 = S_1$,
(2) $S_2S_2 = S_1$,
(3) $S_1S_2 = S_2$.

Thus
$$\begin{aligned} S_1S_1 &= (1+6l_1)(1+6l_2) \\ &= 1+6(l_1+l_2)+6^2l_1l_2 \\ &= 1+6\{(l_1+l_2)+6l_1l_2\} \\ &= 1+6l_3 = S_1 \end{aligned}$$

That is, that in (1) and (2) any number of the first or second series multiplied by any other of the same series as itself always gives a product which is a member of the first series, whilst (3) the product of two members of different series, of the first or the second, always gives one of the second series. All prime numbers come under these rules. (The converse statements, however, are not necessarily true i.e. that $S_1 = S_1S_1$, or $S_1 = S_2S_2$, or $S_2 = S_1S_2$.)

Further it is likewise obvious that

$$S_1S_3 = S_3;\ S_2S_3 = S_3;\ S_3S_3 = S_3.$$

The product of any odd number whether it be a prime or not, by an odd number that is not a prime, is always an odd number that is not a prime since it must contain the factor 3.

N.B.—We use the term 'prime number' here in its generalized sense as meaning any member of the series S_1 or S_2, so that certain multiples of 5, or 7, may be included in the rules, though not strictly speaking 'primes'.

We may lay down the following general propositions:

$$\begin{aligned} S_1S_1 &= S_1, \quad \text{and} \quad & S_1S_3 &= S_3, \\ S_2S_2 &= S_1, & S_2S_3 &= S_3, \\ S_1S_2 &= S_2, & S_3S_3 &= S_3. \end{aligned}$$

Similarly, we have for the three series of even numbers T_1, T_2, T_3,

$$\begin{aligned} T_1T_1 &= T_3, & T_1T_3 &= T_1, \\ T_2T_2 &= T_2, & T_2T_3 &= T_2, \\ T_1T_2 &= T_2, & T_3T_3 &= T_1. \end{aligned}$$

Furthermore,

$$\begin{aligned} T_1S_1 &= T_1, & T_2S_1 &= T_2, & T_3S_1 &= T_3, \\ T_1S_2 &= T_3, & T_2S_2 &= T_2, & T_3S_2 &= T_1, \\ T_1S_3 &= T_2, & T_2S_3 &= T_2, & T_3S_3 &= T_2. \end{aligned}$$

We have also

$$\begin{aligned} T_1+S_1 &= S_3, & T_2+S_1 &= S_1, & T_3+S_1 &= S_2, \\ T_1+S_2 &= S_1, & T_2+S_2 &= S_2, & T_3+S_2 &= S_3, \\ T_1+S_3 &= S_2, & T_2+S_3 &= S_3, & T_3+S_3 &= S_1. \end{aligned}$$

And further

$$\begin{aligned} S_1+S_1 &= T_1, & T_1+T_1 &= T_3, \\ S_1+S_2 &= T_2, & T_1+T_2 &= T_1, \\ S_1+S_3 &= T_3, & T_1+T_3 &= T_2, \end{aligned}$$

$$S_2+S_2 = T_3, \qquad T_2+T_2 = T_2,$$
$$S_2+S_3 = T_1, \qquad T_2+T_3 = T_3,$$
$$S_3+S_3 = T_2, \qquad T_3+T_3 = T_1,$$

These cover the whole range of all possible whole numbers.

Mendel's Law of Heredity

In general, if we take the sum of an even and an odd number, and multiply it by the sum of another pair of an even and an odd, the product is always made up of the sum of three even and one odd.

Thus T and T' being even and S and S' odd,

$$(T+S).(T'+S') = (TT'+TS'+T'S+SS').$$
$$= (T+T+T+S).$$

Now S, S', S'', S''' being four odds,

$$(SS'+S'')(SS'+S''') = (S+S+S+S) = T;\ S \text{ being any odd.}$$

But $$(TS'+S'')(TS'+S'') = (T+T+T+S) = S,$$

T being any even and S any odd.

Furthermore TT' will always produce T's if multiplied by T's or a sum of T's. (We must suppose that $T+S$ is possible only when T is a TS.) Now if we regard T's as *dominant* and S's as *recessive*, the analogy with Mendel's Law of Heredity is apparent (as a combination of groups).

The analogy does not hold unless the T's which do not contain S's as a factor are of the form $T+T$ and multiplied with T's of a similar nature. Provided this limitation is borne in mind, it is obvious that if the dominant characters are represented by an even number and the recessive by an odd, the combinations follow the Mendelean Law.

Hence let even numbers not containing an odd factor represent dominant characters, and odd numbers recessive.

Then we arrive at the Rule:

Even numbers containing an odd factor, if added to an odd, yield by multiplication or association, one even, without odd factors; two even containing odd factors, and one odd, with, of course, no even factors.

Odds if added to odds, always produce by multiplication or association, evens as a sum of odds.

Evens added to evens always produce by multiplication or association, evens.

Evens added to odds produce by multiplication or association, three evens and one odd. One of the evens contains no odd

factor, two evens do contain odd factors, whilst the odd, of course, contains no even factor.

Such, in fact, is Mendel's Law when for 'even' and 'odd' we substitute 'dominant' and 'recessive' respectively.

This is something more than a mere coincidence, and shows undoubtedly that Mendel's Law results from the presence of certain characters or factors in even or odd numbers.

Obviously the numbers may be of any magnitude whatever and may be of the order of millions, billions, or trillions, &c. But the gametes or *chromosomes* contain units of these orders of magnitude.

It would appear that the dominant or recessive characteristics depend upon whether the number is odd or even, without regard to the actual arrangement which may correspond to other characteristics.

In any case we have

$$(T+T)(T+T) = T+T+T+T = T,$$
$$(S+S)(S+S) = S+S+S+S = T,$$
$$(T+S)(T'+S') = TT'+TS'+T'S+SS' = S,$$

and we know that $TT' = T$, $TS' = T$, $T'S = T$,

so that $$(T+S)(T'+S') = T+T+T+S = S.$$

The first T, however, differs from the second and third in that it contains no odd factors, whereas the two latter essentially do so.

We can investigate all the different combinations of S_1, S_2, S_3, T_1, T_2, T_3. Even if all possible combinations had an equal chance to start with, it would appear from the relations given in p. 354, that T_2 predominates, and would, therefore, other things being equal, excel in numbers over all others in the struggle for existence; so that in the long run all other characters would practically be eliminated. The following will therefore outnumber the rest:

$$(T_2+S_1)(T_2+S_1) = T_2+T_2+T_2+S_3,$$
$$(T_2+S_1)(T_2+S_2) = T_2+T_2+T_2+S_2,$$
$$(T_2+S_1)(T_2+S_3) = T_2+T_2+T_2+S_3,$$
$$(T_2+S_2)(T_2+S_3) = T_2+T_2+T_2+S_3,$$
$$(T_2+S_3)(T_2+S_3) = T_2+T_2+T_2+S_3,$$
$$(T_2+S_2)(T_2+S_2) = T_2+T_2+T_2+S_1.$$

The Mendelean Law here holds throughout.

If instead of taking the numbers that give Mendel's Law we substitute for $T = e^P$, and $S = e^Q$, P being the logarithm of an even number and Q that of an odd.

We have then

$$(e^P+e^Q)(e^{P'}+e^{Q'}) = e^{P+P'}+e^{P+Q'}+e^{P'+Q}+e^{Q+Q'}.$$

Now, in general, the logarithms of whole numbers are not whole numbers. If, however, we consider the cases when the logarithms are whole numbers, though the numbers of which they are the logarithms be not so,

then $$(e^T+e^S)(e^{T'}+e^{S'}) = e^{T+T'}+e^{T+S'}+e^{T'+S}+e^{S+S'}.$$

$$= e^T+e^S+e^S+e^T.$$

since

$$T+T' = T, \quad T+S' = S, \quad T'+S = S, \quad S+S' = T,$$

If we choose the base so that the transformation is from whole numbers to whole numbers the characters in the transformed system are equally distributed and not in accordance with the Mendelean Law, but rather like the characteristic law of sex. Hence for a particular distribution of gametes in the cell, their exponential equivalents will be equally distributed. Thus if we regard the characteristic units, whatever they might be, that make up sex, as the exponential equivalents of those characteristics which follow Mendel's Law, the relationship between them will be such that if the units that characterize sex are present in arithmetical progression, then the Mendelean characteristics will be present in geometrical progression. In fact sex bears the same relationship to the other characteristics. The laws governing the two are different but related, by a definite law, and deducible one from the other.

If we take $p^{(a+6m)} = b+6n$, p being the logarithmic base, and $a+6m$ and $b+6n$ the related whole numbers, then for $a(1, 2, \ldots 6)$, $b(1, 2, \ldots 6)$,

$$(a+6m)\log_e p = \log_e(b+6n), \quad \therefore \log_e p = \log_e(b+6n)^{\frac{1}{(a+6m)}}$$

or what comes to the same thing

$$p = (b+6n)^{\frac{1}{(a+6m)}}.$$

This, of course, is possible, where $p = N^{\frac{1}{M}}$, N and M being integers of a particular type, such that

$$N_1^{M_2} = N_2^{M_1}; \; N_2^{M_3} = N_3^{M_2}, \text{ \&c.}$$

For instance $10^{12} = (10^3)^4 = (10^4)^3 = (10^2)^6$,

or $(10^{\frac{1}{4}}) = (10^{3/12})$, &c.

or $(10^{\frac{1}{8}}) = (10^2)^{1/8} = (10^3)^{1/12} = (10^4)^{1/16} = (10^5)^{1/20}$, &c.

This is a particular and simple case.

It is possible, therefore, to find a p such that

$$p = N_1^{1/M_1} = N_2^{1/M_2} = N_3^{1/M_3}, \text{ \&c.,}$$

where N_1, N_2, N_3, &c. and M_1, M_2, M_3, &c., are whole numbers, and not necessarily powers of each other.

$N = (b+6n)$ where $b = (1, 2, 3, \ldots 6)$ and $M = (a+6m)$, $a = (1, 2, 3, \ldots 6)$.

Though it would not hold for all values of b and a, yet there are obviously certain indeterminate values of b and a for which the relation holds. All that we want to know for the present is that it is possible to have a certain base such that certain whole numbers will be represented by it by other whole numbers which are their logarithms according to that base. This being granted, the transformation from sexual gametes to Mendelian determinants becomes clearly possible. Hence it is a feasible procedure to transform exponentially from whole numbers to whole numbers, although the base is neither e nor 10 but p where $p^M = N$, and $M = \log_p N$.

$$M = \frac{\log_e N}{\log_e p} = \frac{\log_{10} N}{\log_{10} p}, \text{ or } \log_{10} p = \frac{\log_{10} N}{M} = \frac{\log_{10} N_n}{M_n}.$$

This indeterminate equation does admit of certain particular solutions which are sufficient for our purpose.

Thus $p = 10$ gives

$$10^1 = 10$$
$$10^2 = 100$$
$$10^3 = 1000$$
$$10^4 = 10000$$
$$\text{\&c.}$$

so that for $M = 1$	$N = 10$	$b = 4+6\times 1$
$M = 2$	$N = 100$	$b = 4+6\times 16$
$M = 3$	$N = 1000$	$b = 4+6\times 166$
$M = 4$	$N = 10000$	$b = 4+6\times 1666$
&c.	&c.	&c.

In Rydberg's Law for spectra arising from the displacement of electrons within the atoms we have $n = R\left(\frac{1}{N_0^2} - \frac{1}{N^2}\right)$, where $R = 3{\cdot}29\times 10^{15}$ and $(N = 2, 3, 4, \ldots)$, n being the frequency of the vibration of a ray of light or other such radiation.

In these if $p^M = N$; $M\log_e p = \log_e N$. so that

$$\frac{d^2M}{dN^2} = C\frac{1}{N^2} \quad \text{or} \quad \left(\frac{d^2M}{dN^2}\right)_0 - \left(\frac{d^2M}{dN^2}\right)_1 = C\left(\frac{1}{N_0^2} - \frac{1}{N_1^2}\right).$$

For particular values $N = 1, 2, 3, 4, \ldots$, therefore, we obtain corresponding values of the second differential of M with regard to small variations on the second degree in N.

There is thus a striking analogy between the constitution of the atom and the cell, for the frequency of the vibrations n in the former may be regarded as equivalent to accelerations of M for the values N_0 and N_1. Just as in the cell, where we have seen there is an exponential connexion between the gametes on the whole and the determinants which group themselves according to Mendel's Law whilst the gametes interchange in proportion to the difference of the reciprocal of the squares of the stimulus, so the nucleus of the atom itself is changed by the process of radiation, through the transition of N in the atom, the proton being altered no less than the position of the electrons. The latter indeed is all that Bohr supposes; but he does not account in any way for the mode of transmission of energy as radiation.

If q is the energy of radiation, and h Plank's constant since $nh = q$, and $q = e^M$,

$$\frac{dM}{dq} = \frac{1}{q} \text{ or } nh = \frac{dq}{dM},$$

that is, the energy of the radiation is proportional to the rate of change of energy with the value of M_1, the quantities corresponding to the determinants in the cell. Thus the radiation is directly proportional to the loss of *some thing* in the atom, represented by M.

This mysterious M is apparently the source of life in matter and is thus closely related to the mechanism of radiation, no less than that of chemical action. The affinity of certain atoms for others thus depends upon the particular radiations they emit which at molecular distances affect each other. At such distances if matter 'attracts' light, light must 'attract' matter, and facilitate chemical action as we know it does.

From p. 348, we have,

$$S_1 = \left(\frac{S_1}{T_1}\right) + \left(\frac{S_1}{T_1}\right)^2 + \left(\frac{S_1}{T_1}\right)^3 + \ldots$$

with similar relationships for S_2, T_2, and S_3, T_3,

Also according to our notation,

$$T_1 = \left(\frac{T_1}{S_3}\right) + \left(\frac{T_1}{S_3}\right)^2 + \left(\frac{T_1}{S_3}\right)^3 + \ldots$$

with similar series for corresponding terms.

Any dominant or recessive gametes may therefore be regarded as made up of an infinite series of the form

$$S_n = \sum \frac{S_n}{T_n}, \qquad T_n = \sum \frac{T_n}{S_{n+1}}.$$

All these transformations are particular cases that come under the theory of Groups in a more general form. What must be borne in mind is that quite apart from the mechanical aspect of things, mere numbers, *i.e.* classes, play a more important part than has hitherto been imagined. And that we have arrived at an explanation of mutation and variations which are retained from scale to scale, microscopic to macroscopic, molecular to terrestrial, magnitudes.

The Law of Sex

Let y be the number of gametes in an aggregate of Sex characteristics and x the number of Mendelian units. Then if, as it is natural to suppose, $\frac{dy}{dx} = k, \left(\frac{1}{x}\right); y = k \log x$, and therefore $y = x^k$.

If T and S are the dominant and recessive characters, let us say male and female (it might be the reverse, female and male, it matters not which as the result will show), we have as in p. 357 e^T and e^S as the aggregates actually involved in sexual phenomena, and thus

$$(e^T+e^S).(e^{T'}+e^{S'}) = e^{T+T'}+e^{T+S'}+e^{T'+S}+e^{S+S'},$$

and since

$$T+T' = T, \quad T+S' = S, \quad T'+S = S, \quad S+S' = T,$$

we get the above equal to

$$e^T+e^S+e^S+e^T,$$

or putting M and F for the sexual characters,

$$(M+F).(M'+F') = M+F+F+M,$$

so that the male and female would occur in equal proportions, provided the constituents followed the Mendelian Law. Thus a close connexion between the sexual and Mendelian characters is established, the male and female aggregates or gametes being exponential functions of the even and odd number of units in the corresponding Mendelian characters.

It is quite obvious that although the Mendelian characters

depend upon odd and even numbers in relation to some unknown units which enter into the operations, this does not apply to sex directly. The point is that if Mendel's Law as the result of the combination of odd and even units in the manner indicated, and larger aggregates of these gametes are formed in such a manner that the rate of increment of the sexual to the Mendelian is inversely proportional to the number of Mendelian units already in the group—a most probable event—then the approximate equality of the sexes in numbers follows from the Mendelian distribution of the units of such characters within the groups forming the sexual gametes. This is one of the most important and practical results of the method of mathematical attack upon such problems.

The important part which the theory of numbers plays in many physical and biological effects has been emphasized, not with a view to returning to any form of Pythagorean mysticism, but to emphasize the necessary relationship that units hold towards each other, and the light the presence of such units throws upon the existence of phenomena attesting the mere presence of these units in combination, according to some definite laws. We have seen that the phenomena embraced under the Mendelian Law of heredity admit of a simple interpretation and coordination with the law of equality of the sexes, by the mere analysis of the units, whether odd or even, so that what was apparently a mystery turns out to be but a result of groups of odd and even numbers.

A similar law should of course hold in chemistry, as in biology. The part played by chemical affinity and valency is so great that such effects as we have been discussing can hardly be expected to predominate. Apart then from the excess of even over odd elements in Nature, and the scattering of α-particles in excess by odd-numbered elements, the tendency is for even-numbered elements to form, and the more stable ones will be even apart from other considerations, for the even and odd relationship applies to the property of stability. So that given a number of stable atoms of a group of elements, the even will predominate over the odd. In other words, the minority among the stable elements are odd.

Now if we examine the relations given in the tables on p. 354, we shall find that whereas all odd numbers are made up of the sum of an odd and even, and never of two odds any more than of two evens, yet even numbers can as readily be made up of two odds as of two evens. Hence, among the evens half or so would contain no even constituents, i.e. no α-particles, which are nothing more than

charged helium atoms, and therefore even. That is to say, that the α-particles are to be found for the most part among the odd elements. This remarkable result follows then, not from the fact that there are more even elements than odd which is not the determining factor in this respect, but the fact that amongst the even a much larger proportion contain no even constituents, and therefore no α-particles.

The question of division opens up still more remarkable ground, in which the theory of numbers reveals many striking phenomena. If we consider the two series S_1 and S_2, since they are composed almost exclusively of the prime numbers $1+6n$, and $5+6n$, it will appear that as prime numbers they contain no factors. Those elements therefore which are not made up of others, if any such exist, are to be found among the odd numbers.

Incommeasurables present an important class of substances. When one group or class will not divide evenly into another, and the division or subdivision can go on indefinitely like a recurring or repeating decimal then something analogous to the phenomenon of reproduction is in sight. This is particularly noticeable when we take into account not only the striking properties in the theory of numbers, already pointed out, but also those of the partition of numbers, as particular cases.

We have seen that in the case of incommeasurables, it is possible for recurring decimals, to take a familiar example, to repeat themselves in groups, indefinitely, and when two such numbers are added or subtracted, the result is also a repeating decimal, see p. 351. Now we may regard all inanimate matter as made up of commeasurable combinations of units, and living matter as composed of incommeasurable combinations or classes which do not exactly divide, and fail to do so in such a way as to render the series of subdivisions continuous. Hence continuity and discontinuity become clearly possible. The theory and partition of numbers shows on the lines we have indicated how it is that reproduction and repetition of identical qualities, and the transmission of combined qualities becomes possible throughout endless generations in the series of time. Nay more, that in space the characteristics in any particular scale of being, are a repetition of those in higher and lower scales. The million, or billion recurring group, repeats itself in that group, from the infinitely small upwards. The oak, which is but the visible manifestation of the inherent qualities of the acorn, contains in the same acorn and stored within its

innumerable monads the history of its past and the prevision of its future. All with unfailing regularity and precision tell the same tale of endless progression both in space and time. The actual transformations are problems of the most abstruse nature. The theory of groups, of algebraic forms, of the partition of numbers, and the theory of equations, of determinants, of covariants and invariants, are involved; all, however, are purely abstract forms dealing with the properties of classes which when applied to matter serve to reveal the inner structure and working of reality, in groups within groups like patterns of logical archetypes, monads in the immense and immeasurable expanse of thought.

The agreements are much too striking to be passed over with sceptical indifference. They only press the question more urgently. To Leibnitz the idea seemed to lurk in an unceasing manner, though he could not at the time clearly see his way through the maze which it presented. But the theory of numbers as a theory of aggregates offers many possibilities, by analogies and nothing more, except when, as in the case of the Mendelian Law of heredity, the more general properties come with greater closeness to our aid, though even then with certain marked limitations. In physics the interaction of the groups presents difficulties which are almost insurmountable. Yet the principle of the Conservation of Energy by the use of the Hamiltonian function, the principle of Least Action enables us to deal with physical problems in a purely generalized manner, as in the case of Lagrange's equations which are amongst the most remarkable illustrations of the possibility of dealing effectively with a vast group of coordinates of which other variables are functions. Sir J. J. Thomson in his *Applications of Dynamics to Physics and Chemistry* has given a number of illustrations of the applicability of the method of generalized coordinates in many branches of Physical Science. The Principle of Energy is assumed as the fundamental basis of his calculations. Such a generalized procedure was perhaps the goal towards which Leibnitz was instinctively aiming, though as the earliest pioneer in the subject he made but little progress in this direction. It was reserved for Lagrange to indicate the way in which to grapple with this problem, and Sir Wm. Rowan Hamilton to enunciate the principle of Least Action. Such limitations as are imposed by the conservation of energy do not enter into non-conservative systems; though Lagrange's equations are then hardly applicable without modification. What concerns us here is the grouping of units. The Conservation of Energy so far as the mathematician or the logician is concerned is a mere equation of relationships.

The Lagrangian equations of motion may be regarded as applying to systems in which the actual nature of the coordinates is unknown, though of course the equations cannot be written down. They can, however, be seen to hold for coordinates which if known would lead to definite results; coordinates of noumena perhaps applied to phenomena. This extension to groups of generalized coordinates would carry us beyond the scope of the present work, and would require a volume to itself.

In his Guthrie Lecture to the Physical Society, on 'Atomic Nuclei and their Transformation', Lord Rutherford directed attention to the results obtained by Harkins who has shown that even-numbered elements are much more abundant in Nature than the odd-numbered elements.

This is as it should be if the elements have been built up of units which were equally distributed to start with. For although odds added to evens produce odds, yet odds added to odds produce evens, and evens added to evens do likewise. Hence at first sight it would appear to be two to one in favour of evens. A glance at the tables on p. 354, will make it quite clear that the overwhelming preponderance is necessarily with the evens in the ratio of 4 : 3, but with multiples of each other the matter becomes more complicated, and the proportion may be as high as 5 : 2, starting with perfect equality. Whether by addition or multiplication the evens always win. Hence Rutherford's striking result that odd-numbered elements which are disintegrated by the bombardment of α-particles emit protons of a much higher average speed than the even-numbered elements, for most odd contain even constituents as α-particles.

It will, of course, be observed that this refers to the purely numerical relations of the atomic weights, and atomic numbers (i.e. preferably called the Moseley numbers). It is a matter of interest, I conceive, to know what and possibly how these effects come about. My object in going to so much trouble in this matter has been to ascertain how far carbon and oxygen might enter into the formation of the elements as a whole. Where multiples of 6, Moseley's number for carbon, constitute the system of classification, carbon might be expected to play an important part in the building up of inorganic, as it does of organic Nature, chiefly on entirely other grounds. The results of this investigation, laborious and thorough as it has been, have led to some curious points of interest. For instance, uranium could be made up of radium and carbon, the emanation Niton or Radon of polonium and carbon,

and radium of polonium and oxygen. Another striking fact is that the emanation is capable of being made up of a very large number of elements in a variety of ways, owing to its particular atomic weight 222. Most of the other combinations are fictitious.

Though most inorganic elements may have a carbon nucleus or be built up from carbon, like all organic ones, there is not sufficient evidence at present for this conjecture, probable as it may seem. It is a property of numbers and yet something more.

CHAPTER XXVIII

THE SYNTHESIS OF LIFE

PHYSICAL Metabolism, or the building up and breaking down of molecular groups, is illustrated in its simplest state by the formation of molecular aggregations by the action of light in phosphorescent substances, notably uranium glass, as shown by the change of absorption of the fluorescent light during the period of fluorescence, a phenomenon discovered by the author in 1896, and communicated to the British Association at Liverpool that year, and later to the Royal Society (*Proceedings* 1897, 1905; and *Philosophical Transactions* 1898). Various observers have since studied this phenomenon. Amongst them R. W. Wood of Baltimore (*Phil. Mag.* Dec. 1908). He denied its existence, having used what virtually amounted to a phosphoroscope which gives the absorption when the body is, and is not, illuminated. But the interval between the two comparisons must be such that the phosphorescence shall have had time to die out completely, else we should be observing the absorption during the period of phosphorescence in either case, and the effect would be the same as Wood found it to be. The phenomenon has, I believe, since been found to take place in many bodies when rendered luminous by fluorescence or electric means indicating that the periods of vibration are newly created during the phosphorescence, of the passage of the electric discharge, and not merely roused up previously existing degrees of freedom, to which Balfour-Stewart and Kirchoff's Law applies. Bodies generally absorb the light they emit. But if there is a change of absorption from the non-luminous to the luminous state, new free periods must be created in the luminous process.*

This colloid molecular aggregation under the influence of light is the first step towards the photosynthesis. Benjamin Moore has synthesized Formaldehyde H—CHO, or CH.OH, but *the presence of uranium is necessary*. This is of very great importance in the synthesis of most organic compounds, beginning with the starches and sugars. There are two kinds of sugars Monoses and Bioses ($C_6H_{12}O_6$) and ($C_{12}H_{22}O_{11}$).

* It may be said that this phenomenon has nothing whatever to do with life; and it certainly has, at first sight, little to do with cellular life. But physical metabolism in colloidal matter is a condition of life and the preliminary and simplest manifestations of life in its physical sense. (See *The Origin of Life* Chapter XIII and XIV.)

Thus the former may be represented in either of the two forms:

Grape sugar glucose (dextrose) was shown by H. Kiliani to be

$(CH_2OH).(CHOH)_4.CHO$, an Aldehyde,

and

Fructose (Laevulose) $(CH_2OH)_2(CHOH)_3CO$, a Ketone.

Either of these is represented by the empirical formula $C_6H_{12}O_6$. Cane-sugar, however, has the empirical formula $C_{12}H_{22}O_{11}$, whilst starch has the empirical formula $(C_6H_{10}O_5)_n$, the actual number of atoms of each element in the molecule being unknown. Yet it is clear that there are twice as many molecules of starch in the molecule of cane-sugar as in the molecule of grape-sugar, and that in either case a molecule of water makes all the difference between the molecule of sugar and that of starch, thus

Grape- and Fructose-sugar $C_6H_{12}O_6 = (C_6H_{10}O_5).(H_2O)$,

and

Sucrose or Cane-Sugar $C_{12}H_{22}O_{11} = (C_6H_{10}O_5)_2.(H_2O)$.

The molecule of cane-sugar may therefore be regarded as consisting of two molecules of starch combined with one of water, whilst that of grape- and fructose-sugar has only one of starch with the same amount of water. Milk-sugar (or lactose) and malt-sugar (maltose) $=(C_{12}H_{22}O_{11}).(H_2O)$ and therefore cane-sugar with a molecular of water $= (C_6H_{10}O_5).(H_2O)_2$.

Now when grape-sugar is fermented, alcohol and carbonic acid are formed; thus

$$C_6H_{12}O_6 = \underset{\text{(Ethyl Alcohol)}}{2C_2H_5OH} + 2CO_2.$$

If then the alcohol be oxidized, starch may be formed, or either of the sugars resynthesized by the action of light or other form of suitable radiation, such as radioactivity, particularly the γ-rays.

The reaction in such a synthesis is

$$3(C_2H_6O)+3O_2 = \underset{\text{(Glucose)}}{C_6H_{10}O_5}+4H_2O,$$
$$= (C_6H_{10}O_5).H_2O+3H_2O.$$

Or we may get

$$6(C_2H_5OH)+6O_2 = \underset{\text{(Fructose)}}{(C_6H_{10}O_5)_2}.H_2O+7H_2O.$$

The general empirical formula for the sugars is $C_n(H_2O)_m$, which may be written

$$\underset{\text{(Starches)}}{C_n(H_2O)_n(H_2O)_{m-n}}, \quad \text{or} \quad C_n(H_2O)_{n-1}.(H_2O)_{m-n+1}.$$

It is of interest to note that grape-sugar can be represented by the formula $(CHOH)_6$, and all sugars by $(CHOH)_n$, or with $(H_2O)_n$ a mere polymeric form of Formaldehyde. And starch as sugar

deprived of a molecule of water thus: $(CHOH)_{n-1}$—C, so that if a molecule of water be taken on by it we obtain sugar thus: It is the intermolecular structure that characterizes the sugar:

$$(CHOH)_{n-1}C.H_2O = (CHOH)_n.$$

The free carbon atom in the case of starch gives it unbounded possibilities in the formation of other organic compounds.

Sugar is also but a polymeric form of Glycolaldehyde

$$CH_2CH.CHO,$$

or else a combination of this and Formaldehyde, thus:

$$C_6H_{12}O_6 = \{(CH_2OH).(CHO)\}_3 = \{(CH_2OH)CHO\}.(CHOH)_4.$$
$$=(CHOH)_6.$$

The molecular weight of $CHOH = C(H_2O)$ and $(CH.OH)_n = C_n(H_2O)_n$.

The synthesis of sugar and starch in this manner by Kiliani and Emile Fischer constitutes a very great advance towards the artificial production of the higher organic compounds which enter into the constitution of living matter, such as Protein. But this has not yet been accomplished, nor do we know what the actual composition of protein is. Yet all the work above throws no light whatever upon the formation and mode of action of the cell, which is largely a question of physics. The study of the mechanism of the cell may, therefore, be approached from the stage where the chemical substances involved are assumed to have been already synthesized. There is little fear of the artificial synthesis not being carried forward. But the investigation of the mechanism of the cell need not be retarded on account of delays in this respect. I have been taken to task by critics who ought to have known better, for postulating the existence of such organic matter in studying the mechanism of the elementary cell. The synthesis must be left to the chemist. But his synthesis will never turn dead matter into living until more than the physical and chemical side of the process is understood. The growth, reproduction, differentiation and behaviour of the nuclei and their chromosomes, the process of karyokinesis, heredity, and so forth, these are problems of a different nature. Not until it is recognized that we have to deal with an entirely different class of phenomena from those usually considered to be within the sphere of physics and chemistry, will new light be thrown upon the subject. And it is with this consideration clearly in view that I have approached the problem in the new light in which it is here presented.

The number of molecules within the nucleus is much too small

to account for the marvellous potentialities and mechanism of the cell and the organism into which it ultimately develops. But by bearing in mind the particular kind of 'time-difference' here proposed, a new class of phenomena presents itself quite distinct from those of spectroscopic analysis. The latter has been recently fully explained by Bohr's theory of the atom. And although difficulties still present themselves in connexion with the mode of emission of radiation by the mere change in the position of an electron within the atom, and also the stability of so complex a structure, the light it throws upon spectral phenomena renders the theory of exceptional value in scientific speculation, at any rate as regards atomic structure. A close affinity may be found between it and the theory of radiation here propounded; for the atom is a cell and what is called a cell a large atom or molecule of colloid matter. Sunlight or electric discharges, or, in their absence, radioactivity, can produce the synthesis of formaldehyde from carbonic acid and thus $CO_2+H_2O = CH.(OH)+O_2$. Fenton has effected the synthesis by metallic magnesium in water saturated with carbonic acid gas.*

From calculations of the size of the living cell Clerk Maxwell concluded that the number of molecules within it cannot exceed two millions apart from the water which the cell contains. When, however, it is borne in mind that the nucleus, to which the vital energy is due, does not occupy more than a very small portion of the cell, the number of molecules is reduced to a few thousands. When, moreover, the *determinants* or *biophors*, namely Weismann's biological units, are reached, the number is reduced to a few hundreds if not to a still smaller number, so that whatever variations the combinations of such units may still admit of—and the almost infinite number of such possible combinations cannot be questioned any more than the almost infinite number of words from the twenty-six letters of the alphabet—*there remains the difficulty of accounting for the mechanism* which can be formed from the juxtaposition of a few such units; since the subsequent development is undoubtedly the result of certain innate qualities in these units of germplasm known as determinants, biophors, or gametes.

* Sir John Murray states that the amount of radioactive substances, and lead their ultimate product of disintegration, of manganese nodules, cosmic spherules, and zeolitic crystals is greatest where the deposition on the sea-floor ought to be least. Professor John Joly of Trinity College, Dublin, has found that the radium-content is larger in deep sea deposits than in terrestrial rocks and that red clay and oozes from deep water contain much more radium than oozes from shallower depths. (See Sir John Murray, *The Ocean.*)

Clerk Maxwell's reasoning is as follows. Two million molecules of hydrogen in a row would occupy a millimetre (that is, the diameter is about 5×10^{-8} cm.), and about two hundred million million million of these would weigh a milligramme, i.e. the mass of an atom of hydrogen is about 5×10^{-24} grs. Nobert was able to draw 4,000 lines in the breadth of a millimeter and the intervals between these lines was observable with a good microscope. A cube whose side is the 4,000th of a millimetre was taken as the *minimum visible*. Such a cube $2{\cdot}5 \times 10^{-5}$ cm. would contain 100 million molecules of oxygen and of nitrogen. But since the molecules of organized substances contain on an average about 50 of the more elementary atoms, we may assume that the smallest organized particle visible under the microscope contains about two million molecules of organic matter. At least half of every living organism consists of water, so that the smallest living being visible under the microscope does not contain more than one million organic molecules. Some exceedingly simple organisms may be supposed to be made up of not more than a million of such molecules. It is impossible, however, to conceive so small a number to be sufficient to form a being furnished with a whole system of specialized organs.* Thus molecular science sets us face to face with physiological theories. It forbids the physiologist from imagining that structural details, unless of infinitely small dimensions, can furnish an explanation of the infinite variety which exists in the properties and functions of one of the most minute organisms.

This of course is assuming as Clerk Maxwell supposed that atoms are indivisible, and, moreover, that such infinitely small structural details could not exist. His calculation shows, in the light of more recent work, that the physiologist's difficulty can be overcome by postulating not only particles smaller than atoms, but the existence of much minuter structures.

'A microscopic germ', he remarks, 'is, as we know, capable of developing into a highly organized animal. Another germ, equally microscopic, becomes, when developed, an animal of a totally different kind. Do all the differences, infinite in number, which distinguish one animal from the other, arise each from some difference in the structure of the respective germs? Even if we admit this as possible, we shall be called upon by the advocates of pan-genesis to admit still greater marvels. For the

* The number of groups that can be formed by a million objects is

$$2^{10^6} = 10^{3 \times 10^5}.$$

This would, no doubt, be quite sufficient for our purpose, as regards possible variations, if that were all that is required. But given a structure of 10^6 atoms the difficulty is to account for the mechanism in any one particular case with only these million atoms as in the case of the sperm or ovum of one of the higher animals. Biologists generally do not seem to appreciate the distinction.

microscopic germ, according to this theory, is no mere individual, but a representative body, containing members collected from every rank of the long-drawn ramification of the ancestral tree, the number of these members being amply sufficient not only to furnish the hereditary characteristics of every organ of the body and every habit of the animal from birth to death, but also to afford a stock of latent gemmules to be passed on in an inactive state from germ to germ, till at last the ancestral peculiarity which it represents is revived in some remote descendant.

'Some of the exponents of this theory of heredity have attempted to elude the difficulty of placing a whole world of wonders within a body so small and so devoid of visible structure as a germ, by using the phrase *structureless germs* coined by Sir Francis Galton. Now one material system can differ from another only in the configuration and motion which it has at a given instant. To explain differences of function and development of a germ without assuming differences of structure is, therefore, to admit that the properties of a germ are not those of a purely material system.'*

This theory of pan-genesis requires that every cell in the body of the organism should be represented by a gemmule in the germ-cell. The body-cells dispatch their representatives to the germ-cell, which is the seat of the gemmules. Darwin laid much stress upon this, but it has now been abandoned by most biologists as unnecessary. Yet the outstanding difficulty remains the same, on the germplasm theory. It is idle to endeavour to overcome the difficulty by attributing memory to germplasm. This may be dismissed on physical grounds, for the reasons given herewith.

Biologists, like Hartig and Samuel Butler, who hold that acquired habits are transmitted and transmitted by such memory, are willing to postulate some *plasticity* in the molecules. This of course would be a very simple way of overcoming the difficulty if acquired habits be really transmitted. As it is, however, the more orthodox biologists, such as Weismann, do not admit that acquired habits are transmitted, except perhaps very slowly.

The latter regard the potential qualities as stored up in the biological units, and only an infinite variety of modes of motion within these would explain their infinite potentialities. But dynamical theories do not enable us to form a picture or model of how these infinite varieties of motion can take place within the molecule or atom, unless it is made up of infinitely minute and indefinitely numerous particles.

Postulating almost infinitely minute particles as it is quite justifiable so to do, the condensation of their aggregates into spiral nebular forms of innumerable shapes, which ultimately resolve themselves into atoms and electrons, is not only thinkable but

* 'Atom', Clerk Maxwell's *Collected Works*.

probable if the analogy holds, particularly in view of the modern wave-mechanics. That being premised the biological atom or molecule can take its place beside the chemical molecule, atom, and electron. No one has ever yet succeeded in isolating these biological units, but their existence is unquestionable on dynamical considerations and as the result of Clerk Maxwell's calculations. Maxwell, it is true, did not offer any explanation of the difficulty, nor was any explanation forthcoming at the time, as he assumed that 'the atoms were manufactured articles and indestructible'. Granting, however, the formation of atoms and electrons from the aggregation of nebular spiral forms, the infinite variations and properties of the atoms is at once apparent.

The difficulties presented are, I think, overcome by the time-change vibration proposed in earlier chapters and the idea of space put forward, since the time-function of the monad eliminates the obstacle of size and number of particles, transforming the whole problem into one-time variations or differences in the manner already indicated.

CHAPTER XXIX

ULTRA-SPATIAL OR NON-SPATIAL REALITY

IT is admitted that the world is a reality, but only a reality as a system of ideas. Material objects exist, and exist independently of our minds, but only as ideas. They do not exist in space relations, except when perceived by our minds, and then only through our organs of perception. The universe, as we perceive it, is limited to space-relations and by time in a space-time continuum. This is a system of monads approximately in syntony out of an infinite host. And a change in the tune, so to speak, of time-relations could transform us into another, a better or a less harmonious grouping of monads constituting another Universe or another world, from which we may or may not perceive the events of this so-called material Universe. Each world is like a lantern-slide, illumined, penetrated, or reflected by the light of the mind, the intellectual lamp itself, to whose variability then the individual atoms of the picture on the screen respond. And the possibility of the individual responding to more than one world at a time depends upon his range of apprehension or octaves of perception; just as two pictures might be thrown simultaneously on a screen, one through a red, and another through a blue screen, and either be seen distinctly if viewed through correspondingly coloured glasses. There are minds that undoubtedly perceive events imperceptible to ordinary individuals. Whatever scepticism may be entertained upon this point, the facts are to my mind, and to many of the most competent observers of our time, well established. Notwithstanding the trickiness and deception of charlatanry, there can be no shadow of doubt that certain phenomena of clairvoyance in hypnotized media are true objective facts, even if such facts have not yet been recognized by the scientific world as scientific, owing perhaps to their *a priori* improbability and our inability to make them fit into any coherent system of thought as yet held by truly scientific-minded men. But the veil is gradually being lifted, and new worlds are coming into distinct vision, and the principles of Mathematical Philosophy draw into a focus the landscape of reality to a consistent and coherent whole.

I do not propose to consider, at the stage of the investigation we have reached, whether this new theory offers a solution of a more satisfactory nature than others yet proposed of ultra-physical or metaphysical problems, or even familiar psychological ones. Whether, for instance, the 'spiritual light' of no less a modern

thinker than Rudolf Eucken is or is not a phenomenon with which the physicist in view of these considerations should concern himself. What I do emphasize is that there is nothing in Physical Science as here viewed in any way antagonistic to it. Nay more, the principles laid down embrace some of the highest conceptions of transcendentalism and to some extent lift the veil of mysticism which for centuries has enshrouded the human mind, as Tut-ank-ahman's tomb concealed a Pharaoh.

It would be out of place to approach these higher mysteries in a purely scientific and philosophic investigation. One cannot expect to soar higher than others who have ventured upon the same flight with futile effort and supersensuous speculation. But the course here pursued seems full of promise; and though the promised land may ne'er be attained, the heights reached, the land traversed cannot on careful scrutiny be claimed to be purely imaginary and fictitious.

The truths and principles thus reached are both scientific and philosophic, and full of suggestiveness towards a spiritual aspect of life. If 'a thousand years are as a day; and a day as a thousand years' or a thousand miles as one, and one mile as a thousand, the work of reconstruction has been well begun. And Science with her elder sister Philosophy might well turn over a new page, and commence a new chapter in the history of human thought, reading side by side the great book of life, animated with a fresh conception of space and a firmer grasp of reality, with hopes of progress in a new direction.

The solution which I now offer is of particular interest, for it overcomes many difficulties usually regarded as utterly inexplicable, but at the same time firmly held by many earnest and profound thinkers as common-sense unquestionable 'spiritual truths', without which life itself would become meaningless as a senseless pandemonium. When 'The Heavens are opened, and we seem to see God Himself', as Bishop Barnes would say, after the manner of St. Paul at his conversion on the road to Damascus, we are conscious of a reality that is not in the space order, but none the less manifest irrespectively of the bodily senses. This is the true spiritual light of the theologian and the mystic, of whose presence the man of science usually, though not always, but with reason, feels sceptical. His scepticism is due to the apparent irreconcilability of such a belief with the materialistic doctrines of the physical science of the nineteenth century and before, though not perhaps of that after. Being broader in our vision we can well divest ourselves not only of the idea of the ether, but still more readily, as

it would now appear, of that space that has so long deluded our path and obscured the true vision of reality.

Whether, indeed, such an apparently fantastic notion, at any rate to the scientific mind, as the resurrection of the body, the most wonderful and I should say almost incredible mystery of theological teaching, one of the most astounding tenets of the Christian Faith, insisted upon by St. Paul, and firmly held even to this day by some of the most earnest and devout scholars of our time, would ever admit of a scientific explanation, seems to me in the light of this consideration to be not altogether eliminated from the class of possible and rational beliefs. Absurd as it may seem, to take an extreme case, that the atoms that composed the body of a person who had been shattered to smithereens by an explosion in the trenches should by any possibility ever recombine, except by what would truly be called a miracle, there is nothing in the light of these reflections to hinder the monad from adopting the right 'time' and 'tune' that would enable it once more to respond to one and all of the monads that had served as the chamber of its material environment. 'Resurgam: non omnis moriar' becomes the hopeful gospel of the man of science as he awakens and perceives this illuminated reality evolving or emerging from the limbo of his once well-founded incredulity.

The aspiration to immortality, and still more indeed to resurrection, may seem to be almost childish in its expectation, in the light of ordinary conceptions of the material world. To most educated persons the Buddhistic idea of annihilation appears more in accordance with a higher nature that bows to the inevitable. Like Napoleon as he embarked upon the *Bellerophon* to end his days as a prisoner at St. Helena, so did Marcus Aurelius look upon death and the grave.

But prejudice one way or the other should have no place in our minds. Only the calm cold light of reason should prevail. And whether it is nobler to be absorbed in the Nirvana, free from pain and suffering and all pleasure too, or to be destined and condemned, for better or for worse, to a life eternal, the reality and the truth are a question of 'so or not so' and not of whether one would or would not have it so. The life eternal is now more than a possibility, to some perhaps a pleasurable, to many a painful reality, both now and for evermore. There is no reason to suppose that this is the worst of all possible worlds, though it might be the best.

If ideas be regarded as monads in relation to which self-conscious monads form a perspective, truth becomes tinged by the individual aspect and the relative importance we attach to the various ideas

with respect to each other—a theologian by the theological aspect, a scientist by the scientific facts. Thus a Christian may be carried away by the spiritual aspect of Christianity, the sceptic by the historical facts in their nakedness, without regard to religion, some by the Church, others by the Bible. Each isolates himself in a world in which the predominant functions are the spiritual or the material. Likewise in politics, the Liberal regards everything from the point of liberty, individuality, Free Trade, maximum wealth, and so forth; the Conservative from the point of view of security, stability, authority, and the welfare of the state on the whole before that of the individual, and so forth. The man educated from a particular point of view is almost hypnotized by it, prejudiced by his perspective, and seldom considers that the fact that he was born and reared in a particular atmosphere of thought has nothing whatever to do with the probability of its being true. His prejudices are natural, and so the monk in a monastery lives in a world quite different from that of the man in the street, in the luxury or the din of the traffic in Park Lane. The imagination, or the lack of imagination, controls the perspective. The question is with Pilate, 'What is Truth?' One might regard it as pragmatic, were it not inevitably self-contradictory. A higher synthesis may be attained if we regarded all these as particular aspects. The truth itself is transcendental and no individual can see it at once from all points of view. Each lives in his own particular world.

There may, however, be more than a mere change of perspective. The monad may, through the influence of certain monads around it or most closely associated with it, respond and act upon different sets or groups of monads; so that its so-called environment may change, as in passing from one state to another. The permanence or constancy of its relations with the environment can be tested only by the congruency with the beliefs of others; for error or illusion, the play of the imagination and inaccurate use of reason can be answered only by consistency and coherency and the estoppel which reason itself imposes on the basis of thought. The theosophist may be right, but unless he can verify his conclusions by scientific tests, he can convince nobody except himself, and those who have like experiences, that he is not labouring under some delusion. Scientific method including mathematical analysis is the only safe test of truths, though it is possible from personal experience to hold beliefs that do not admit of definite proof. Natural selection has enabled the normal healthy individual to feel assured that a thing is so and not otherwise, to such a high

degree of probability or moral certainty that to act or think otherwise would be contrary to his nature. And yet this life may be but a sort of dream from which we might wake up into a higher world and more glorious vision of reality, our bodies and nervous systems here forming merely a part of our environment in the world at present around us. Those in the habit of 'listening in' will most readily appreciate the adequacy of this conception. Only experimental tests will enable us to decide between truth and error. That the brain is akin to a telephone or broadcasting apparatus which we carry about with us is an hypothesis that cannot be overlooked. The change of tune and the 'capacity' of the monad would transform it into other worlds as harmonious as our own here 'below'. The appearance of life affords evidence of other worlds than ours in which like ideas are developed.

The time-differences being a function $f(t)$ that varies from the average or standard, if we like absolute time, gives the monad a characteristic value which determines its environment relatively to itself. The monads in this world or universe have all certain characteristics, and the growth of living matter is due to the relative change in the fundamental note as one gains on the other in relative size and complexity in space.

Only monads of a particular type respond to these individuals that whether in their fundamentals or partials are so attuned, and the world which constitutes the class of monads, of myriad numbers no doubt, represents the special group adjustable to the particular symphony, if we might so call it, the majestic concert of creation. But if these in the Platonic sense possess some such characteristics as self-existent monads, then Truth itself becomes a matter of sympathetic resonance manifesting itself as a 'perspective'. A coherent view of things spiritual may be held irrespectively of things material, apparently inconsistent with it, as if the soul were lifted into another sphere of being. And the mystic for this time, as it were, experiences relations with spiritual realities while unaware of the existence of his relations with the material world, as the musician is carried away in ecstasies of refined perception, unconscious of the presence of his body while contemplating the glories of the sublime in that music which for the time being conveys to him the meaning of the harmony of the eternal. The emergence of life in a particular environment such as Nature is itself a manifestation of the change in group of a monad through some special combination or grouping of ideas.

We are now once more confronted with the same familiar obstacle that impeded our path, and obstructed the reasoning of all in-

telligent beings since man first learnt to take life seriously and attempted to view it coherently or to view it whole. Not merely the origin but also the destiny, the last end of things, now concerns us. Not only the Why? or the Wherefore? the Whence? or the How? But likewise the Whither? and still more the final Purpose and true Aim of it all! A few million centuries will see the end and futility of Man's greatest triumphs and of all his proudest achievements. Silent, and as it were unconscious, will he lie in the bosom of 'space', and still nurtured by time, if time and space indeed could then have any meaning; when the very Earth itself shall have returned to the dust of atoms from which it was once evolved; and Man its former ruler rests wrapped in everlasting sleep as a grain of sand in the infinite desert of the Eternal! When Prospero's prophecy will have been fulfilled 'leaving not a rack behind'. 'Immortal' Shakespeare, Milton, Plato, Newton will be no more, not only utterly forgotten but all their works permanently erased, as mortal as their mortal selves! in fact indeed 'as though they never had been!' The Paradise thus lost, if matter as we now know it has its sway, will never be regained, unless indeed Truth is but falsehood seen from another side; and Reality, the rival of 'hateful error, melancholy's child', is but Nature inside out; the Conjugate Universe to that we now inhabit. Divine philosophy can show to the apt thoughts of men, not only the things that are not, but the things that are. When the theory of the 'dissipation of energy' shall have itself been dissipated, as perhaps it has already been, and the perpetual presence of the monad established and affirmed, who knows or dare assert that in the transformed scale of being his former self, when space might be no more, shall not retain its identity, unconscious of the change, fresh with the full vigour of immortal youth in worlds no less wonderful, nor ideas less sublime in those transcendental regions of Reality, where magnitude is of no consequence, and all things move merrily as of yore in an Absolute Idea. Matter which then will perhaps 'know itself no longer in the long lapse of time', may still be known by mind which knows it now, where it should likewise cease to be in the annihilation of space and in the cessation of time by the transcendental and real world of the ever permanent. Well did Cato soliloquize to his own soul in Addison's beautiful words:*

Plato, thou reason'st well; else whence this pleasing hope,
This fond desire; this longing after immortality?
Why shrinks the soul back on itself,
And startles at destruction? 'Tis the Divinity,

* Slightly modified to suit the occasion.

That stirs within us, 'tis Heaven itself that points out an hereafter
And intimates Eternity to man!
Eternity! Thou pleasing dreadful thought!
Through what a variety of untried beings,
Through what new scenes and changes must we pass!
Here will I hold, if there 's a power above us,
And that there is all Nature cries aloud through all His works,
He must delight in Virtue. And that which He delights in must be happy!
Why then? The stars shall fade; the very sun himself grow dim with age,
And Nature sink in years. But thou and I or all of us will flourish in immortal youth,
Unhurt amid the war of elements, the wreck of matter, and the crash of worlds!

This idea that that which perceives and recognizes the Universal, as in the recognition of Virtue and of Knowledge, the highest of the Virtues, and their performance, must itself transcend the ideas of its own perception which it can control and mould according to its own desire, is of unique value, and constitutes the very essence of the Platonic Theory of Ideas, that so many 'would-be critics' fail to understand. I myself have been the victim of an onslaught in the *Saturday Review* for daring to make use of so 'consoling an argument' in the *Origin of Life* (p. 79). And yet the substance of this present work, as the result of twenty years of impartial study and reflection on the subject, has not only confirmed but strengthened the sentiments I then expressed as a young man, perhaps then only with innate though unsubstantiated conviction.

But even if all monads as we know them ceased to be, there would still be that infinite monad, without which the whole would be unmeaning and a mere empty dream. The denial of the existence of an Infinite Being renders reality or the belief in reality as a rational assent impossible, and converts being into a mere solipsism, worthy of children and primitive peoples, whose natural instincts have, perhaps, enabled them to conquer the belief as a practical issue, by adopting in outline what the very highest flights of the human intellect inevitably confirms as the basis of its own pursuits.

Space, almost like truth itself, though subject to the individual's particular perspective is nevertheless an idea in the infinite monad. The theory of relativity is thus not violated but confirmed by it, and Nature appears once more as a true manuscript of God.

We have been compelled to abandon the idea insisted upon by Leibnitz that all monads are absolutely independent, and that only harmony enables them to see things alike. Monads, like every other object or class of ideas, must overlap; if not here, then there or somewhere, i.e. at infinity, just as any two curves of whatever shape soever even if they seem to be mutually exclusive, as in the case

of two circles that do not cut or touch each other, yet intersect at infinity, and all classes of ideas do likewise.* Even monads do the same, so that we perceive all men to be in truth united as assured us by the Christian teaching, even as 'He is in the Father and the Father is in Him!' All things are united, however much they differ, for every class of ideas—since a class can be represented by a closed curve which distinguishes the class from everything else not of that class; and another class likewise, these two classes being mutually exclusive, so that the two closed curves neither cut nor touch each other—must none the less intersect at infinity. Fantastic as this notion may seem, for a transcendental conception of space is still invoked to get over the difficulty, yet must it be borne in mind that the geometrical analogy applicable to any kind of space, though not essential, serves for the sake of elucidation, and has been so introduced, as the argument is strictly algebraic. As fully pointed out in Chapter V, the totality involves the negation of the premisses without which they are not true.

The *whence*, *whither*, and *why* are thus seen to be foolish and meaningless questions. *Die Welt ist Alle was der Fall ist*, says Wittgenstein, 'The World is all that is the case, or that is so.' To be 'In tune with the Infinite', as Mr. Trime puts it, that is the practical view of life. And, further, if all *negative* ideas be reversed and turned into *positive* ones; to think rightly, feel rightly, and act rightly, and have the right attitude of mind, disposition, state of consciousness, and thus be in tune with the Infinite and in harmony with the Universe, namely one's environment, is the great and highest end. To act in all things nobly, truthfully, sincerely, and courageously, is the highest ideal to which intellect and character amongst civilized men can aspire. It is the religion—the wisdom of the ages—to-day that the highest education and culture should instil. There is no name for it: whether with Spinoza, or with Goethe, or with Kant. God the Infinite and the transcendental truth in the finite and the practical, that is the philosophy as it is the religion of Wisdom. 'Wisdom is thus greater than Knowledge.' 'Get wisdom,' says the Psalmist, 'and with all thy getting get understanding.' It is the clear perception of the Infinite, the great work of Mathematics during the past thirty years, that has paved the way to the understanding of the great mystery of being. To be in harmony with one's natural

* I must admit that the idea of *reducibility of types* which, I understand, has been abandoned by the authors of *Principia Mathematica* scarcely meets the difficulty of self-consciousness itself. And in fact it does not appear to be essential to the principle laid down in that great work, as Ramsey and others have pointed out.

self, confident, self-sufficient, though not self-complacent; to enjoy and derive full pleasure from the best in life; 'to make existence satisfactory', in Goethe's words, that is life itself and true culture. In Art we add to Nature that art which Nature in us makes; it is an art that reveals Nature, although it be the art itself in Nature, which we but perceive, for the two though different are ultimately one.

There is no justification for the supposition that Science and Letters should not move hand in hand; in fact literary culture without mathematical acumen may be said, as of old, to be 'the abomination of desolation standing in the sacred place' of Reason, spoken of by the Prophet Daniel, as the desecration of the Temple of Knowledge by the worship of idols and images instead of the Reality with which we are in actual contact, the Idea itself. We are each one of us in the Real Presence and should know it, as when we commune one with another through the link of Truth: that profound, permanent Reality that underlies our Being and brings us in harmony with each other, reader and author as one, in the Idea we contemplate in common.

CHAPTER XXX

THE PLATONIC IDEALS

MODERN thought has made great strides, but at the same time developed marked retrogressive tendencies which the requirements of to-day serve not only to retain but to intensify at the expense of some of the finer attributes of human nature, upon which it exercises a crushing influence. Its effect indeed in outline is twofold; for where there has been progress in material enterprise there has too been retrogression in idealistic thought; and the centre of inertia of ideas, rough-hew it as we may, would appear to remain fixed like that of the universe itself. We are in truth notwithstanding new emergences all that the past has made us, and find ourselves, with our predilections, sentiments, habits, customs, superstitions, and beliefs, the very slaves, much as we may endeavour to disown the fact, to those conventionalities of thought and prejudice which the traditions of our civilization have bequeathed to us, though some of the fundamental ideas are true. And this temperament affects, however indirectly, our most modern conceptions in scientific thought. Inextricably entangled as we find ourselves in a skein of fallacies obscure and confused, they appear withal to most of us as the very amenities of our existence, to which as human beings we still may cling, if not with timid at any rate with fond tenacity, as the ivy to the oak, though the oak stands firm against the storm.

In truth it cannot be asserted that the historic ideal of material progress, which since the Renaissance with an ever-increasing grasp has taken a firm hold on the minds of men, has in a like measure served to clear our ideas, or to add to our happiness as to the hopes of the future or even served to ennoble our conduct at the present time. Material progress has facilitated man's powers over the physical forces that envelop him to a degree which has excelled the brightest hopes of the inductive reformers of the period that has witnessed the greatest revolution in the perspective of Nature.

It is a proud reflection at times that to no country does the world owe more for this great change in the outlook of mankind than to our own, and that to no three individuals is it more indebted than to Francis Bacon, Sir Isaac Newton, and Charles Darwin. For the Inductive Philosophy owes its inception to the first, its birth and its growth to the second, and, as a practical method of investigation, its realization to the third. Yet, curiously enough, this

inversion of men's thoughts and their relation to the Universe has served to concentrate their attention upon themselves, instead of emphasizing their own insignificance and unimportance; and in the affairs of life to direct their energies to its comforts and its ease as the one great object and end of existence; whilst calculations directed to that purpose have become the incessant efforts of the most active amongst us, not strictly in accordance with their own ideals.

Ideals thus developed, useful as they may seem, possess merely a particular type of utility. And this same material progress has been accompanied by a no less remarkable idealistic retrogression.

If all that glitters is not gold, so all that gold can make or buy is but an unworthy dross of those nobler elements which lie hidden in the human mind and heart. The fighting man of the past has been supplanted by others of the present, whilst the development of the superman, not Nietzschean it is to be hoped, can only be expected with the arrival of still newer methods. This is what the Baconian philosophy and the utilitarian principles of modern times have served more than anything else to reveal and by their methods to retard, if not to crush in the evolution of the Higher Self. Fastidious, some may say, is this perpetual cry, yet does it not remind us that mere pleasure is not always good, and that our dissatisfaction with the hedonist principle is due in part to the overwhelming power of its extravagant pretensions. Our confidence is checked by methods of exchange: dealing as these do in cash values, at a considerable discount, with much that we know will be of real value to mankind not to-day, perhaps, nor to-morrow, but in years to come, though little credit is given for it, as compared with the grosser current notions whose immediate values happen to be more apparent.

These utilitarian principles have been carried into the sphere of scientific inquiry. We have not only pragmatists but scientists with 'a policy'.

The 'arm-chair philosopher' is a term of reproach in the laboratory, where it is not always realized that the use of one's brains is as important as the manipulation of one's fingers even in practical studies. And experimental researches, like material enterprises, have engrossed our care so much that attention has been diverted from the path of philosophic studies. Yet all men who think, or even profess to think, must at every time of life give utterance to philosophic opinions, unconscious as they may be of the fact and great as may be their contempt for all that borders on philosophy. There is no commoner cry than that So-and-so is a philosopher, a

dreamer, and therefore unpractical; or that some one else is a metaphysician, and that he thinks, if not too clearly, at least too hard: that thinking is after all futile or that if practicable at least beyond their heads, forgetting all the while in their own opinions that they are only giving utterance to views or beliefs which they themselves admit to be beyond their own proper spheres, but upon which they appear to be ever ready to argue and perhaps forfeit their lives if not their very souls.

Modern studies and modern methods have, no doubt, worked wonders, and done much to uproot the accustomed habits of thought: to free man from the bonds of ignorance, credulity, and childish fancies which before the era of criticism he was wont to accept as 'facts'. The great transformation from the *a priori* to the *a posteriori* methods of thought, from the deductive to the inductive, exercised a profound influence upon the subsequent trend of English and of European thought.

But the inductive method of reasoning was not altogether new. It was the method that all men employed since they began to reason, and it had been worked out by Aristotle as a branch of formal Science in the *Ὄργανον* (300 B.C.) from which Bacon's *Novum Organum* derives its name.

In the evolution of ideas, as in the ways and habits of the individual, a stage is reached when it would be folly to persist in one direction farther, without pausing a while to consider if we are really on the right path. Sterility in results, as I have said, breeds discontent if nothing else as a Hegelian antithesis or Irish bull. And when such a phase is reached it is felt that if not better, not worse results would follow from trying something new. That appears to have been the state of mind of many at the time of Bacon, and of none more so than of Bacon himself. It is the state of mind of many to-day. The question now is whether, great as the results have been, we have not lost once more the true perspective of Nature, and are expecting from it far more than it can be made to yield: for empirical knowledge is not all knowledge, nor is it even possible without the mind that perceives it, a point which the student of Nature does not always realize. The Hegelian Dialectic once more forces a still higher synthesis.

In its proper aspect the path to be pursued appears to lie not only in the interpretation of Nature and the discovery of her ways; not merely in the knowledge of the laws which regulate the outer world, or with the details generally conceived to be within the province of Science, such as the development of bone, or the minute anatomy and habits of the crab and the butterfly. Our

concern is not so much with Nature or man as an organism as with man as mind and the spirit of man as a reality.

It so happens, however, that the Empirical Philosophy has become the philosophy of the 'man in the street', and that as the man of science complains of the crude ideas of the former about science, particularly pure science, so the philosopher is equally entitled to complain that the views of the man of science or the man in the laboratory about matter and man and the nature of man are on a level with those of his pedestrian antagonist.

There is a faculty wanting, lost or undeveloped—that is all. There is no lack of sincerity, perhaps, but there is some lack of sense.

This book, then, is in part an attempt to justify the Platonic theory of ideas in the light of modern research and the Leibnitzian theory of monads as metaphysical realities or ideas of which the world and truth are composed. Comparing this philosophy with that of Bacon, Macaulay cynically remarks that 'Every trace of intellectual cultivation was there, except a harvest'.

'If the tree which Socrates planted and Plato watered is to be judged by its flowers and leaves it is the noblest of trees. But if we take the homely test of Bacon, if we judge the tree by its fruits, our opinion of it may perhaps be less favourable. . . . We are forced', he maintains, 'to say with Bacon that this celebrated philosophy ended in nothing but disputation, that it was neither a vineyard nor an olive ground but an intricate wood of briers and thistles, from which those who lost themselves in it brought back many scratches and no food.' This indeed is scarcely true. There was no smut or stubble in the elevation of the ideals of conduct and in the victories which represented the triumphs of man over brute creation. The fruits could only minister to the stomach, but the blossom, the flowers and the leaves were the true fruits of the mind. In every attempt to extol the Baconian at the expense of the Platonic ideal those who speak in this manner betray the crudest enthusiasm for an exaggerated and not very lofty notion. The objection may have been applied to all learning previous to the time of Copernicus, in the same manner in which it is made use of to-day to deride the pursuit of those problems which still remain unsolved. Like the study of astronomy and other less economical pursuits, philosophy must not be despised, for the questions it raises are more difficult than those of ordinary science and therefore not so easily answered. Though 'a question properly asked', as Clifford once remarked, 'may be said to be half answered'.

If the Baconian principles have done anything towards the

solution of the problem which philosophy presents, it is to show that its methods are unable to cope with them and that hope of progress in such directions must be sought on some other than Baconian lines.*

It may in truth be safely stated that, great as has been the progress in material welfare since the Renaissance, a progress in which, particularly in recent years, this country has taken a leading part, the steps in other matters have been mainly retrogressive. Locke, Hobbes, Butler, Bentham, Mill, added little or nothing to metaphysical thought, and it is upon Berkeley and Hume, one an Irishman and the other a Scotsman, that the claim of the British people as an idealistic and philosophic nation mainly rests.

It is, however, to the credit of these two men that with the exception of Descartes and Leibnitz, who preceded them, modern philosophy is based upon their work, notwithstanding the claim put forward by Bacon's most ardent admirer that he was the Bonaparte, not the Robespierre, of the great intellectual revolution of the sixteenth and seventeenth centuries.

This attitude is merely critical, but the true function of criticism is to see things as they are or as reason requires them to be, assuming always that the reasoning is not fallacious.

The aim of the Platonic philosophy, it is said, was to exalt man into a god, which is scarcely true; the aim of the Baconian to provide him with what he wants while he continues to be a man, which is perhaps nearer the truth.

Now the object of the Greek philosophy was to teach man to live up to the highest ideals of art in reason and in morals, in literature, in music, in sculpture, and in all else that may concern him.

If its object was to teach man to bear with pain and sorrow when inevitable, and to make life an art according to the best ideas of the time, then a god let him be: for no higher aim can be presented to him. The Baconian philosophy does not detract from it, it is true, but provides for the attainment of this high ideal, by enabling man to dispense with the necessaries so as to pave the way for the luxuries of his existence. But when so provided, the highest end should be the cultivation and appreciation of the Ideal of every description. For human Genius is the realization of the idea, the Logos put into words and deeds, which needs be opened like a flower that blossoms from the seed in its beauty of perfection.

Permutation and combination of a few units, as, for instance,

* This view seems also to be held by Dr. Whitehead in his *Process and Reality.*

the letters of the alphabet, will account for the indefinite number of varieties, just as we may form therefrom an infinitude of words. But the mechanism and heredity admit of no such simple explanation, and we are compelled to fall back upon more subtle consideration.

In the contemplation of Art we are driven to the very depths of creation as well as the nature of truth and the structure of Reality. 'Deeper than the rhythm of Art', says Mr. Theodore Watts-Dunton, 'is the rhythm which Art itself would fain catch, the rhythm of Nature, for the rhythm of Nature is the rhythm of life itself.'

We feel instinctively, and that instinctive feeling is fully attested on scientific grounds by what is urged in this book, that there is something within us, a something that we do not fully appreciate and understand, which none the less when fully aroused serves to make life itself real and beautiful to all who would but perceive it so, a link between us and the rest of things, a bond of union holding us to Nature and to God. 'Our thoughts are deeper than we can fathom', says Emerson. Each is a universe of ideas in himself and these ideas like himself are the ultimate realities.

'In all beauty', says Bacon, 'there is some strangeness of proportion': and it is a curious fact that the rhythmic forms which regulate the configuration of electrons with the atom; and those of the planets, the analogues, as we have seen, of the atomic solar systems in their curious harmonic groupings, in the proportions 3, 4, and 5, or 16, 20, and 24; that these rhythmic harmonic relationships are precisely those which appear to guide the sense of beauty in architecture, in music, in poetry, in our style in prose and speech as well as in every action, revealing the mysterious depths of reality in the monad and ideas in themselves.

From the time of Aristotle to our own, men have wondered what this mystic power could be. A law of thought perhaps, through some rhythmic combination of the elements, in the nuclear substance of the soul in germ-plasm; some aggregate of mind-stuff units which have come together somehow and have contrived to move thus, this way and not otherwise, but which by their modes of operation seem to give an intelligence to Nature and the structure of reality; in truth, as we have seen, the rhythmic scintillation of time-differences between the monads themselves has been the mystery.

This parallelism between the rhythmic laws of Nature and of thought needed explanation, and falls in most admirably with the idea of time-waves. It is an idea which runs throughout Nature

in dealing with the structure of the monad. By its means we seem to attain that reality which is the essence of the sublime and beautiful in the contemplation of all works of Art.

There are few who will deny that for some reason or another the advance in such matters has been slow, and that it is not impossible that we may still be groping in the dark because we have not managed as yet to get hold of Nature in the right way. Dr. Faustus enthusiastically exclaimed, 'Where can I grasp thee, infinite Nature, where?' The answer is surely 'in the inmost recesses of the human mind and soul'.

A posteriori methods of thought are of importance in dealing with empirical knowledge. But the problem to be dealt with primarily and essentially involves the great enigma of reality. Such progress as may be expected in this line of inquiry is to be sought not merely by inductive methods, which appear already somewhat to have failed, but by reverting to the *intuitive a priori* methods of analysing the elements of human knowledge. It must be fully realized that the foundation of all knowledge is the solid ground of man's own inner consciousness, and that as the servant of Nature he is at the same time Nature's most determined master. The Ego, conscious unit and will power, which surveys, reflects, reacts, controls, and comprehends the organism of countless other units composing in aggregate himself, is a unit which realizes truth, and figures space as but a form of thought, and time as a mere mode of it. Into these two moulds of thought all knowledge of Nature must be cast ere it can become an object of experience.

The immaterial essence of thought which is the basis of all knowledge, not of Nature merely, but of all experience, even of the supernatural, of man as man and not mere animal or beast, this pure and lofty essence of being which he discovers in the contemplation and the critical examination of his own faculties, is still a closed book to many, as it was an unwritten book to Bacon. It is to Leibnitz we owe the first great step in this direction and to Kant the recognition of transcendental perceptions.

The materialists do not understand the nature of the problem to be attacked, and keep revolving in a circle like white mice, deriving no doubt much pleasure and amusement from their pursuits, but making no progress in the spiritual direction save that of the little circle of their own narrow and limited experience.

The results of our inquiry then are threefold. In the first place they show the inadequacy of the received or accepted molecular theory in its present form, of the atomic and electron hypothesis

without extrapolation, to explain the mechanism of germ-plasm, nay, that it is not only insufficient but that it cannot do so. In the second place we are led to put in its place the hypothesis of monads, such as Leibnitz supposed, without, however, making use of the principle of pre-established harmony. Room is thus made for evolution as a necessary outcome of the new Idealism. And in the third we have recourse to the principle of 'time change-waves' as applied to the monadology and epistemology, in relation to the theory of emergence of both life and mind in the so-called material Universe. We are in touch with Reality itself as an Idea, and know the world and each other as Ideas in the presence of this great Reality.

On the principle of continuity, to avoid contradiction, all ideas, as monads, are conceived to be realities and self-existent, all ideas bearing some trace of consciousness; our very words, in writing and in speech, being conscious in a proportionate degree. The whisper of the gale to the Aeolian harp or 'the Sicilian shepherd's pierced and jointed reeds' speak for themselves not only of the music which we hear but of the soul of the music itself. In music and in thought we are in tune with reality, for it is the Idea itself that is perceived, as we perceive one another as spirits and commune in thought.

The notion further may be novel but it is, I think, true. If we are, as all modern idealists maintain, such stuff as dreams are made of, then unless we agree with Hume that all is a panorama and that there are no other conscious units but ourselves, the essence of the idea is consciousness. This is the alternative to the dream theory or solipsism. It is an extension of the Monadology of Leibnitz to the Platonic Theory of Ideas. As Leibnitz supposed all atoms to have a trace of consciousness, we suppose all ideas, not only to have an independent reality, but to be endowed with some degree of consciousness. That is the result to which consistency and continuity of thought on Leibnitz's lines would lead. Atoms and all ideas are merely monads, and therefore conscious ideas. The essence of existence is not only to be perceived with Berkeley, but in however remote a degree to perceive with Leibnitz or *to be* with Plato. That only is existent which perceives and is perceived. And the permanent possibility of sensation, which Mill regarded as the cause of our sensations, must carry with it, though to an indefinitely small degree, some trace of that self-consciousness which is itself a perception, else it would be non-existent. The perception of truth is this resonance or concert of ideas.

Some certainly stand for self-conscious beings, that is, other beings than ourselves. Thus a consistent idealism would lead one to infer that not only those ideas which we call monads and atoms are self-conscious but all ideas, and that these are self-existent. This generalization brings the Platonic theory in harmony with Leibnitz's notions, and places both high towards the pinnacle of human thought, and through the aid of the mathematical philosophy towards further development of intellectual feats. But the work is hardly yet begun. And if life on Earth will last for yet a thousand million years, as there is reason to suppose it will, it is not unlikely that man may yet realize the purpose of his existence, 'while memory holds a seat in this distracted globe', and till mankind passes completely into the conjuge universe of negation. For the world is still young. We may outlive it all in growth, in beauty, and in strength in worlds more real though yet unknown, worlds of clearer apprehension and emergent growth unknown to-day but not unknowable in the aeons that are yet to come.

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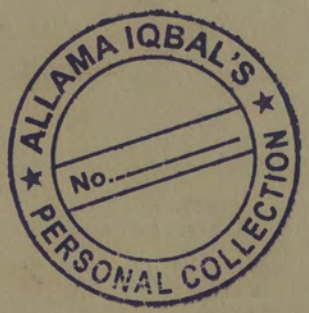

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